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On the smooth linear section of the Grassmannian $Gr(2,n)$

by

Fei Xu

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APPROVED, THESIS COMMITTEE:

Brendan Hassett,
Professor of Mathematics, Chair

Andrew Putman,
Assistant Professor of Mathematics

Wotao Yin,
Assistant Professor of
Computational and Applied Mathematics

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Abstract

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In this thesis, we will study the smooth linear section of the Grassmannian $Gr(2, n)$. Explicitly, we give a criterion for the rationality of such linear section in terms of its codimension in the Plücker embedding in projective space. Moreover, to obtain a better understanding of the birational parametrization of these linear sections, we analyze their Hodge structures in the cases of even and odd codimensions. To be more precise, we provide numerous examples which suggest certain patterns of Hodge diamonds corresponding to even and odd cases and derive the proof of general patterns for codimension 3 smooth linear section of $Gr(2, n)$ corresponding to odd and even n.

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To my parents.

Contents

Chapter 1

Introduction

1.1 Historical background

Let X be an algebraic variety over K. We recall the definition of the rationality of X.

Definition 1.1. A variety X is **rational** if there is a birational morphism over K,

$$X \dashrightarrow \mathbb{P}^n_K.$$

We can also say that X is birationally equivalent to projective space $\mathbb{P}^n_K$.

It turns out that the question of which varieties are rational is one of the subtlest geometric problems one can ask. The project of determining which varieties are rational (or not) has led to the development of a large amount of rich and beautiful theory. On a historical note, there are classical results for low dimensional cases. A necessary and sufficient condition for an algebraic curve of arbitrary degree over $K = \bar{K}$ to be rational is given by the Cayley - Riemann criterion: *a curve is rational* *if and only if the genus of the curve is 0.* The rationality problem for quadratic and cubic surfaces was settled over one hundred years ago, that is, all irreducible quadratic and cubic surfaces are rationally parametrizable (except the elliptic cubic cylinders or cones). As the dimension goes higher, the project to determine the rationality gets harder. V. A. Iskovskikh and J. Manin [1] showed a smooth quartic threefold is never rational by demonstrating the birational automorphism group of the smooth quartic is finite. H. Clemens and P. Griffiths [2] proved that smooth cubic threefolds are not rational by identifying a cohomological obstruction, arising from the intermediate Jacobian.

For many years, mathematicians have worked on the rationality of smooth complete intersections, but currently very few of them are known to be rational. Moreover, there are no smooth projective hypersurfaces of any degree $d \geq 4$ that are known to be rational [3]. In his thesis, M. Reid [4] showed that the nonsingular intersection of two quadrics $Q_1 \cap Q_2$ is always rational. Explicitly, in an odd dimensional projective space $\mathbb{P}^n$, namely, $n = 2g + 1$ for some positive integer $g > 1$, the intermediate Jacobian of the general complete intersection is isomorphic (as a principally polarized abelian variety) to the ordinary Jacobian of the hyperelliptic curve which is the double cover C of $\mathbb{P}^1$ branched over the $n + 1$ points corresponding to the singular quadrics in the pencil of quadrics through $Q_1 \cap Q_2$. The birational mapping $\mathbb{P}^{n-2} \dashrightarrow Q_1 \cap Q_2$ blows up this curve. In this case, the Hodge diamond of the complete intersection is:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & \iddots & & & & \vdots & & & & \ddots & \\
0 & & 0 & & g & & g & & 0 & & 0 \\
 & \ddots & & & & \vdots & & & & \iddots & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

We see that the middle cohomology classes coincide with the first cohomology class of C, which is actually inherited from the latter.

1.2 Hodge structures

In this section, we recall the Hodge-Deligne structure of algebraic varieties over $\mathbb{C}$. Let H be a finite-dimensional complex vector space. A *pure Hodge structure of weight k on H* is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, the bar denoting complex conjugation in H. (A Hodge structure is usually defined over the field of the rational numbers, meaning that we have a vector space H over $\mathbb{Q}$ whose complexification $H_{\mathbb{C}} = H \otimes \mathbb{C}$ is endowed with a Hodge decomposition as above. Since we shall not need this, we limit ourselves to using Hodge structures over $\mathbb{R}$.)

The k dimensional cohomology $H^k(X)$ of a compact Kähler manifold X (in particular, an algebraic smooth projective variety) has a natural Hodge structure of weight k; here we write $H^k(X)$ for the cohomology with complex coefficients. We denote

$$h^{p,q}(X) = \dim H^{p,q},$$

which is called the Hodge number of type (p,q). A Hodge structure of weight k on H gives rise to the so-called *Hodge filtration* F on H, where

$$F^p = \bigoplus_{s \geq p} H^{s,k-s},$$

which is a descending filtration. Note that $\mathrm{Gr}_F^p H = F^p/F^{p+1} = H^{p,q}$.

The consequence for the *Betti numbers* is that

$$b_k = \dim H^k = \sum_{p+q=k} h^{p,q},$$

where the sum runs over all pairs p, q with $p + q = k$. The sequence of Betti numbers becomes a diamond of Hodge numbers spread out into two dimensions. Introduce the *Euler characteristic*

$$\chi(X) = \sum_k (-1)^k b_k(X).$$

Let H be a complex vector space. A *(mixed) Hodge structure* over H consists of an ascending weight filtration W on H and a descending Hodge filtration F on H such that F induces a pure Hodge filtration of weight k on each $\mathrm{Gr}_k^W H = W_k/W_{k-1}$.

Again, we define

$$h^{p,q}(H) = \dim H^{p,q}, \qquad \text{where } H^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H.$$

Given a manifold X, we let $A_c^k(X)$ be the real vector space of k-forms on X with compact support, and d the standard exterior derivative. Then *the de Rham cohomology groups with compact support* $H_c^q(X)$ are the homology of the chain complex $(A_c^\bullet(X), d)$:

$$0 \to A_c^0(X) \to A_c^1(X) \to A_c^2(X) \to \cdots,$$

i.e., $H_c^q(X)$ is the vector space of closed q-forms modulo that of exact q-forms.

They also demonstrate contravariant behavior with respect to proper maps - that is, maps such that the inverse image of every compact set is compact([5]). Let $f : Y \to X$ be such a map; then the pullback

$$f^* : \Omega_c^q(X) \to \Omega_c^q(Y) : \sum_I g_I dx_{i_1} \wedge \cdots \wedge dx_{i_q} \to (\sum_I g_I \circ f) d(x_{i_1} \circ f) \wedge \cdots \wedge d(x_{i_q} \circ f)$$

induces a map $H_c^q X \to H_c^q Y$.

If T is a submanifold of X and $U = X - T$ is the complementary open set, there is a long exact sequence

$$0 \cdots \to H_c^q(U) \to H_c^q(X) \to H_c^q(T) \to H_c^{q+1}(U) \to \cdots \tag{1.1}$$

called the long exact sequence of cohomology with compact support([5]).

P. Deligne [6] has shown that, for each complex algebraic variety Z, both the cohomology $H^k(Z)$ and the cohomology groups with compact support $H_c^k(Z)$ carry natural mixed Hodge structures, with weight filtration W and Hodge filtration F, which coincide with the classical (pure) Hodge structure when Z is a smooth projective variety.

Serre duality is a duality present on non-singular projective algebraic varieties X of dimension n (and in greater generality for vector bundles and further, for coherent sheaves), i.e., $H^q(\Omega_X^p) \cong H^{n-q}(\Omega_X^{n-p})^\vee$. If the variety is defined over the complex numbers, this yields a refinement of Poincaré duality, which relates H^i to H^{2n-i}, considering X as a real manifold of dimension $2n$.

In the case for holomorphic vector bundle E over a smooth compact complex manifold X, the statement takes the form:

$$H^q(X, E) \cong H^{n-q}(X, K \otimes E^\vee)^\vee,$$

in which X is not necessarily projective and K is the canonical line bundle.

1.3 Schubert cycles and Pieri's formula

> The key to the garden of geometry on Grassmannians is in the hands of an old guard, the Schubert cycles. (R.Donagi [7])

A *Grassmannian* is a space which parametrizes all linear subspaces of a vector space of a given dimension. Roughly speaking, a Grassmannian is a generalization of projective spaces - instead of looking at the set of lines of some vector space, we look at the set of all planes of certain dimension. For example, the Grassmannian $Gr(2,n)$ is the space of 2-planes in an n dimensional space.

Definition 1.2. Let V be an n dimensional vector space over a field K. The **Plücker embedding** realizes the Grassmannian $Gr(k,n)$ as a closed subvariety of the projective space $\mathbb{P}(\wedge^k V)$ of the k-th exterior power of the n dimensional space, namely, $Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ taking $< v_1, \cdots, v_k >$ to $[v_1 \wedge \cdots \wedge v_k]$.

We will devote this section to the Schubert cycles which form a natural basis for the integral cohomology and give a cell decomposition of $Gr(k,n)$, with cells only in even dimensions, all realized by (singular) algebraic subvarieties.

A *Schubert variety* is a certain subvariety of a Grassmannian, usually with singular points. Described by means of linear algebra, a typical example consists of the k dimensional subspaces V of an n dimensional vector space W, such that

$$\dim(V \cap W_j) \geq j$$

for $j = 0, 1, \cdots, k-1$, where $W_0 \subset W_1 \subset \cdots \subset W_{k-1}$, $\dim W_j = n - k + j - a_j$ is a certain flag of subspaces in W and $n - k \geq a_0 \geq a_1 \geq \cdots \geq a_{k-1}$. The significance of Schubert varieties is that the cohomology of the Grassmanian, and more generally, of more general flag varieties, is spanned by the cohomology classes of Schubert varieties, which are named the Schubert cycles.

Definition 1.3. We define the **Schubert cycle** $\sigma_{a_0,\cdots,a_{k-1}}$ to be

$$\{A \in Gr(k,n) | \forall i, \dim(A \cap W_i) \geq i\}.$$

It is straightforward to check that codim $\sigma_{a_0,\cdots,a_{k-1}} = \sum_{i=0}^{k-1} a_i$. One convention we make here is that $\sigma_{a_0,\cdots,a_l,0,\cdots,0} = \sigma_{a_0,\cdots,a_l}$.

Next we analyze some important Schubert cycles of the Grassmannian:

- σ_1 is a (singular) hyperplane section of $Gr(k,n)$;
- σ_{n-k} consists of all subspaces containing a fixed point;
- $\sigma_{n-k,n-k,\cdots,n-k}$ (l times) consists of subspaces containing a fixed $\mathbb{P}^{l-1}$; in particular, for $l = k$, $\sigma_{n-k,n-k,\cdots,n-k}$ is a unique point.

What we will use to carry out the computation in $H^*(Gr(k,n))$ in this thesis is the following formula of Pieri:

$$\sigma_{a_0,\cdots,a_{k-1}} \cdot \sigma_a = \sum \sigma_{b_0,\cdots,b_{k-1}}$$

where the summation goes over all k-th tuples $(b_0, \cdots, b_{k-1})$ satisfying

$$n - k \geq b_0 \geq a_0 \geq b_1 \geq a_1 \geq \cdots \geq b_{k-1} \geq a_{k-1},$$

and

$$\sum_{i=0}^{k-1} b_i = \sum_{i=0}^{k-1} a_i + a.$$

Remark: Just to emphasize, we point out a special case of Pieri's formula $\sigma_{a,b} \cdot \sigma_1 = \sigma_{a+1,b} + \sigma_{a,b+1}$; here $\sigma_{p,q} = 0$ unless $n - 2 \geq p \geq q \geq 0$.

1.4 Hodge structures of the Grassmannian and its smooth linear sections

Inspired by Reid's results [4], in this thesis I analyze the rationality of the smooth linear section of the Grassmannian and its Hodge structure. The Hodge diamond sheds light on what subvarieties are blown up in the birational parametrization. Throughout, we assume the ground field to be the field of complex numbers $\mathbb{C}$. All terms of a topological nature refer to the usual complex topology.

The Grassmannian itself has a relatively easy Hodge structure. For example, in $Gr(2, n)$, all the nontrivial cohomology classes are generated by algebraic cycles which are closures of affine space, which shows that $h^{p,q} = 0$ if $p \neq q$. The Schubert cycles determine the Hodge diamond of $Gr(2, n)$, that is, when $p = q$, we have $h^{p,p} = \#\{(a, b) | n - 2 \geq a \geq b \geq 0, a + b = p\}$, more explicitly,

$$h^{p,p} = [\frac{p+1}{2}] = 1, 1, 2, 2, 3, 3, \cdots \text{ if } p \leq n - 2.$$

Note that $\dim Gr(2, n) = 2(n - 2)$, so $H^{n-2,n-2}$ is in the middle cohomology. And by symmetry of Hodge diamonds, the cohomology beyond the middle degree is completely determined by those below the middle degree (simply reflect the upper half of the Hodge diamond across the middle degree).

We denote $X_{r,n}$,or X for short, as a smooth linear section of the Grassmannian $Gr(2,n)$ of codimension r in its Plücker embedding in projective space, where $r \leq 2(n-2)$. The Lefschetz hyperplane theorem says $h^{p,q}(X) = h^{p,q}(Gr(2,n))$ when $p+q < \dim X$. Therefore the part of the Hodge diamond of X above the middle line is the truncated Hodge diamond of $Gr(2,n)$. As a result, only the middle row needs to be completed separately, since those numbers can not be determined from the Hodge numbers of $Gr(2,n)$.

1.5 Summary of results

In this thesis, we study the linear section $X_{r,n}$ defined above. To appreciate how our results fit into the literature, we divide our work into the following three parts: Theorem 2.2.1 in Chapter 2 gives the rationality criterion of $X_{r,n}$ and the proof is given in Section 2.3; the pattern of Hodge diamonds of $X_{r,n}$ is stated at the beginning of Chapter 3, while the proof of the main theorem 3.1.1 is given in Section 3.3; Chapter 4 develops standard machinery for computing Hodge diamonds of complete intersections, which can be applied to more general cases other than smooth linear sections of the Grassmannian; we give a few examples to illustrate higher codimension cases in Section 4.2, Hodge diamonds that have not been computed in any other way; we also demonstrate some special cases of Theorem 3.1.1 in Section 4.3 by actually going through the computation of all Hodge numbers. More importantly, the computation gives the Hodge diamond of $X_{3,8}$, which is needed in the proof of the main theorem 3.1.1 as a missing case.

Chapter 2

The rationality of the smooth linear section of $Gr(2, n)$

2.1 Classification of Fano varieties

Throughout, we assume the ground field K to be the field of complex numbers $\mathbb{C}$.

In order to get prepared for section 2.2, we introduce the well-known classification theory for Fano varieties first.

Definition 2.1. A **Fano variety** is a smooth projective variety V whose anticanonical line bundle is ample.

In particular Fano varieties all have Kodaira dimension $-\infty$. By the Kodaira vanishing theorem, the Hodge numbers $h^{p,0}(V) = h^{0,p}(V)$ are zero for $p \neq 0$.

The simplest examples are obtained by taking smooth complete intersections of hypersurfaces of degrees $(m_1, m_2, \cdots, m_k)$ in $\mathbb{P}^n$. By the adjunction formula, such a complete intersection is Fano if and only if $\sum_i m_i \leq n$. Rational homogeneous varieties G/P (G is semisimple, P is parabolic) are Fano too([8]). And the Grassmannian is contained in this category.

A Fano curve is, obviously, $\mathbb{P}^1$. If V is Fano of dimension 2 then V is called a *Del Pezzo* surface. Such surfaces have been classically studied, and it is well-known that any such V is isomorphic either to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ blown up in d points $(1 \leq d \leq 8)$ in general position; general position means here that no three points are on a line, no six points on a conic, no eight points on a cubic with a singularity at one of the points. For $1 \leq d \leq 6$, the anticanonical map is an embedding. It realizes a blow-up of $\mathbb{P}^2$ in d points $(1 \leq d \leq 6)$ as a surface V_l of degree $l = K_V^2 = 9 - d$ in $\mathbb{P}^l$. For $d = 7$, one obtains V_2 which is a double cover of $\mathbb{P}^2$ ramified along a quartic, and for $d = 8$, a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$. Del Pezzo surfaces are again all rational.

In dimension three, there are non-rational examples (we will see an explicit example in section 2.2). V. A. Iskovskih ([9]) classified the Fano threefolds with second Betti number one into 18 classes, and S. Mori and S. Mukai ([10]) classified the ones with second Betti number $b_2 \geq 2$, finding 87 deformation classes.

The *index* of a Fano manifold V is defined as the maximal integer such that the canonical sheaf K_V is divisible by that integer in $Pic(V)$. The index number is actually well-defined for any smooth variety, but below we mention a few generalities on Fano manifolds:

Theorem 2.1.1. *([11]) Let V be a Fano n dimensional variety. Then the index* *ind(V) is at most $n+1$; moreover, if ind(V) $= n+1$ then $V \cong \mathbb{P}^n$, and if ind(V) $= n$ then V is a quadric hypersurface.*

Remark: $ind(V) \leq n+1$ follows immediately from bend-and-break; but in fact the result of S. Kobayashi and T. Ochiai in [11] is much older, and the proof of their first statement is quite elementary.

The Grassmannian is Fano as a rational homogeneous variety. We can also check this from the definitions. We take the following short exact sequence

$$0 \longrightarrow R \longrightarrow \mathcal{O}_{Gr(k,n)}^{\oplus n} \longrightarrow Q \longrightarrow 0,$$

where R and Q are defined to be the universal subbundle and the universal quotient bundle of $Gr(k,n)$ respectively; we know that $T_{Gr(k,n)} = Hom(R,Q)$. Because $\sigma_1 = c_1(Q) = -c_1(R) = c_1(\mathcal{O}_{Gr(k,n)}(1))$,

$$\begin{aligned} -K_{Gr(k,n)} &= c_1(T_{Gr(k,n)}) \\ &= rank(R)c_1(Q) - rank(Q)c_1(R) \\ &= k\sigma_1 + (n-k)\sigma_1 \\ &= n\sigma_1. \end{aligned}$$

We now return to the smooth linear section $X = X_{r,n}$ of the Grassmannian $Gr(2,n)$ in the Plücker embedding. It is not hard to find that the index of $X_{r,n}$ is $ind(X) = n - r$, since the adjunction formula gives the canonical bundle $K_X = \mathcal{O}(-n+r)$.

2.2 Main theorem and examples

In general, $K_{X_{r,n}} = \mathcal{O}(-n+r)$ gives the following statements:

- if $n < r < \frac{n^2-n-2}{2}$, X is of general type;
- if $r = n$, X is Calabi-Yau;
- if $r \leq n-1$, X is Fano.

Moreover, we also have the following rationality theorem:

Theorem 2.2.1. *$X_{r,n}$ is rational when $r \leq n-2$.*

Remark: The $r = 2$ case has been extensively studied previously. Theorem 2.2.1 tells that $X_{2,n}$ is rational for any reasonable n; in terms of the Hodge structure, not only complete intersections, but actually all smooth subvarieties of $Gr(k,n)$ of codimension 2 have been studied by W. Barth and A. Van de Ven [12]. So we will focus on $X_{r,n}$ with $r \geq 3$.

Before we give the proof of Theorem 2.2.1, we look at some examples to illustrate the theorem:

Example 2.2.2. Look at $X_{3,5} \subset Gr(2,5)$, $K_X = \mathcal{O}(-2)$. By Iskovskih's classification, $X_{3,5}$ is a Fano 3-fold with index 2. We also have the intersection number

$$\mathcal{O}(1) \cdot \mathcal{O}(1) \cdot \mathcal{O}(1) \cdot X_{3,5} = \mathcal{O}(1)^6 \cdot Gr(2,5) = \deg Gr(2,5) = 5.$$

Then by the classification theory of Fano 3-folds, we know $X_{3,5}$ is birational to a quadric in $\mathbb{P}^4$, which is rational (Theorem 4.2 of [13]).

Example 2.2.3. Take $r = 4, n = 5$; $X_{4,5}$ in $Gr(2,5)$ is a Fano surface, since $K_{X_{4,5}} = \mathcal{O}(-1)$.

$$K^2_{X_{4,5}} = X_{4,5} \cdot \mathcal{O}(-1) \cdot \mathcal{O}(-1) = Gr(2,5) \cdot \mathcal{O}(-1)^6 = \deg Gr(2,5).$$

Note that the Hilbert polynomial of $X_{4,5}$ is

$$\chi(\mathcal{O}(m)) = \frac{5}{6!}m^2 + \text{lower order terms},$$

this gives $\deg X_{4,5} = 5$.

In conclusion, $X_{4,5}$ is a Del Pezzo surface of degree 5. And hence by classification of Del Pezzo surfaces, $X_{4,5}$ is given by blowing up 4 points in $\mathbb{P}^2$ with no 3 points on a line, therefore it is rational.

Example 2.2.4. We look at linear sections $X_{r,6}$ in $Gr(2,6)$ for various values of the codimension r. Note that the dimension of $Gr(2,6)$ is 8, so the codimension r varies from $3 \leq r \leq 7$. When $r = 3$ or 4, Theorem 2.2.1 tells us that $X_{3,6}$ and $X_{4,6}$ are both rational.

K. Takeuchi [14] and V. A. Iskovskih [15] constructed two different birational maps from $X_{5,6}$ to a cubic threefold. M. Noether proved that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational [16]. But as we mentioned in introduction, H. Clemens and P. Griffiths [2] showed that cubic threefolds are not rational, by using Hodge theory and intermediate Jacobian.

For $X_{6,6}$, the Euler characteristic $\chi(X_{6,6}) = c_2(T_{X_{6,6}})$. By the adjunction formula, we know that $K_{X_{6,6}} = 0$; and Riemann Roch tells us that

$$\chi(\mathcal{O}_{X_{6,6}}) = \frac{c_2(T_{X_{6,6}}) + K^2_{X_{6,6}}}{12} = \frac{c_2(T_{X_{6,6}})}{12}.$$

It then suffices to compute $\chi(\mathcal{O}_{X_{6,6}})$. As $X_{6,6}$ is simply connected and connected by the Lefschetz hyperplane theorem, we get $h^0(\mathcal{O}_{X_{6,6}}) = 1$ and $h^1(\mathcal{O}_{X_{6,6}}) = 0$. Meanwhile, $K_{X_{6,6}} = 0$ gives $h^2(\mathcal{O}_{X_{6,6}}) = 1$. That is how we see that $\chi(\mathcal{O}_{X_{6,6}}) = 2$, and hence $c_2(T_{X_{6,6}}) = 24$. So we conclude that $X_{6,6}$ is a K3 surface.

The last case is when $r = 7$, adjunction formula gives $K_{X_{7,6}} = \mathcal{O}_{Gr(2,6)}(1)$. $X_{7,6}$ is a curve of genus $g = 8$.

In conclusion, the rationality of the linear section X is presented below:

r	linear section $X_{r,6}$
$r \leq 4$	rational variety
$r = 5$	birational to a cubic threefold, unirational but non-rational
$r = 6$	a $K3$ surface, trivial canonical bundle
$r = 7$	a curve of genus 8, positive canonical bundle

2.3 Proof of main theorem

Proof. We work over a field K of characteristic zero. Fix an n dimensional vector space V. We can regard $Gr(2, n) = \{P \subset V | \dim P = 2\}$. Then we fix an $(n-1)$ dimensional subspace $W \subset V$.

For a generic choice of P as above, we have

$$\dim(P \cap W) = 1$$

for dimension reasons.

Therefore, we can construct a rational map φ as follows:

$$\varphi : Gr(2,n) \dashrightarrow \mathbb{P}W = \{\text{lines in } W\} = \mathbb{P}^{n-2}$$
$$P \longrightarrow P \cap W$$

For any line $l \in \mathbb{P}W$, the fiber

$$\varphi^{-1}(l) = \{P \in Gr(2,n) | P \cap W = l\},$$

which is exactly the Schubert cycle σ_{n-2} and itself is a plane $\mathbb{P}^{n-2}$ in the Plücker embedding.

Now if X is a generic linear section of $Gr(2,n)$ of codimension $r \leq n-2$, we have

$$\varphi|_X : X \dashrightarrow \mathbb{P}W = \mathbb{P}^{n-2}$$

whose fiber $\varphi^{-1}|_X(l) = \sigma_{n-2} \cdot \sigma_1^r$.

Since σ_1 represents a hyperplane section in the Plücker embedding, $\sigma_{n-2} \cdot \sigma_1^r = \mathbb{P}^{n-2-r}$. Hence X is birational to a $\mathbb{P}^{n-2-r}$-bundle over $\mathbb{P}^{n-2}$, which is furthermore birational to $\mathbb{P}^{2n-4-r}$. In conclusion, X is rational for $r \leq n-2$.

Remark: When the codimension $r = n - 1$, $X_{r,n}$ is still Fano, but the above rationality proof fails for this case. Example 2.2.3 provides a good example for $X_{4,5}$ to be rational under this circumstance. The other side of the story is illustrated in Example 2.2.4: $X_{5,6}$ is not rational but unirational. It seems fair to ask if $X_{r,n}$ satisfies any weaker rationality condition for the $r = n - 1$ case, considering we do not have any counterexample for which $X_{n-1,n}$ fails to be unirational. It is an open problem whether codimension $(n-1)$ sections are unirational.

Chapter 3

Hodge diamond of smooth linear sections of $Gr(2, n)$

3.1 Main theorem

As we mentioned in chapter 1, the stratification of the Grassmannians and Lefschetz Hyperplane section theorem give all the Hodge numbers of the linear section X except the middle row of the Hodge diamond. We have the precise pattern for the following case:

Theorem 3.1.1. *(a) When $r = 3$ and $n = 2m + 1$ is odd, the Hodge diamond of $X_{3,2m+1}$ has no nontrivial middle cohomology classes. The Hodge diamond is of the pattern:*

$$
\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&1&&&&&&&&&&& \\
&&&&&&&&&&0&&0&&&&&&&&&& \\
&&&&&&&&&0&&1&&0&&&&&&&&& \\
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&&&&&&&0&&0&&2&&0&&0&&&&&&& \\
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&0&&&&&&&&0&&m-1&&0&&&&&&&&0& \\
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&&&&&&&0&&0&&2&&0&&0&&&&&&& \\
&&&&&&&&0&&0&&0&&0&&&&&&&& \\
&&&&&&&&&0&&1&&0&&&&&&&&& \\
&&&&&&&&&&0&&0&&&&&&&&&& \\
&&&&&&&&&&&1&&&&&&&&&&&
\end{array}
$$

(b) When $n = 2m$ is even, the Hodge diamond of $X_{3,2m}$ is of the pattern:

$$
\begin{array}{c}
1\\
0 \qquad\qquad 0\\
0 \qquad 1 \qquad 0\\
0 \quad 0 \qquad\qquad 0 \quad 0\\
0 \quad 0 \qquad 2 \qquad 0 \quad 0\\
0 \quad 0 \quad 0 \qquad\qquad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \qquad 2 \qquad 0 \quad 0 \quad 0\\
\\
\cdot^{\cdot^{\cdot}} \qquad\qquad\qquad \vdots \qquad\qquad\qquad \ddots\\
\\
0 \qquad\qquad\qquad\qquad m-1 \qquad\qquad\qquad\qquad 0\\
0 \qquad \cdots \qquad \binom{m-1}{2} \qquad\qquad \binom{m-1}{2} \qquad \cdots \qquad 0\\
0 \qquad\qquad\qquad\qquad m-1 \qquad\qquad\qquad\qquad 0\\
\\
\ddots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \cdot^{\cdot^{\cdot}}\\
\\
0 \quad 0 \quad 0 \qquad 2 \qquad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \qquad\qquad 0 \quad 0 \quad 0\\
0 \quad 0 \qquad 2 \qquad 0 \quad 0\\
0 \quad 0 \qquad\qquad 0 \quad 0\\
0 \qquad 1 \qquad 0\\
0 \qquad\qquad 0\\
1
\end{array}
$$

3.2 Hodge type of subvarieties of the Grassmannian

Recall that $X_{r,n} \subset Gr(2,n)$ is a complex subvariety which is defined by sections $f_1, \cdots, f_r$ with $f_i \in H^0(Gr(2,n), \mathcal{O}(d_i)), d_1 = \cdots = d_r = 1$. Denote the complement of $X_{r,n}$ as Z. One defines the *Hodge type of Z* to be the largest a for which the Hodge-Deligne filtration $F^\bullet$ on the de Rham cohomology of Z with compact supports satisfies

$$F^a H^i_c(Z) = H^i_c(Z) \ \ \forall i.$$

Theorem 3.2.1. *(Pedro L. del Angel R.[17]) The Hodge type of $Z = Gr(2,n) - X_{r,n}$ is at least*

$$\kappa = m - r + 1,$$

where $m = n-1$ if $n \neq 8$; $m = n-2$ when $n = 8$.

Using Theorem 3.2.1, we get the following corollary:

Corollary 3.1. *When $n \neq 8$, the middle cohomology classes of $X_{3,n}$ all vanish except for the middle two.*

If we apply the long exact sequence of cohomology with compact support (1.1) in Section 1.2 to $X = Gr(2,n), T = X_{r,n}, U = Z$, then we get

$$\cdots \to H_c^i(Z) \to H^i(Gr(2,n)) \to H^i(X_{r,n}) \to H_c^{i+1}(Z) \to \cdots .$$

When $r = 3$, $\dim X_{3,n} = 2n-7$ and $H^{2n-7}(Gr(2,n))$ vanishes, we have

$$0 \to H^{2n-7}(X_{3,n}) \to H_c^{2n-6}(Z) \to H^{2n-6}(Gr(2,n)) \to \cdots ,$$

which makes the Hodge structure map from $H^{2n-7}(X_{3,n})$ to $H_c^{2n-6}(Z)$ injective.

Recall that a morphism between two Hodge structures H, H' is a complex linear map $f : H \to H'$ compatible with both the weight and Hodge filtrations. The most relevant fact is that morphisms of Hodge structures are strictly compatible with the filtrations W and F, i.e.

$$f(F^\bullet(H)) \subset F^\bullet(H'), f(W_\bullet(H)) \subset W_\bullet(H').$$

Therefore, the above exact sequence preserves Hodge structures.

If $n \neq 8$, Theorem 3.2.1 gives $H_c^{2n-6}(Z) = H_c^{n-3,n-3}(Z)$, meanwhile $H^{2n-6}(Gr(2,n)) = H^{n-3,n-3}(Gr(2,n))$. We therefore conclude that $h^{p,q}(X_{3,n}) = 0$ for $p+q = 2n-7$ except for the middle two hodge numbers.

Remark: Theorem 3.2.1 is not sharp enough to deduce Theorem 3.1.1 because it misses the following two points:

(1) when r is even, there is no such injective Hodge structure map between $X_{r,n}$ and Z, therefore we cannot derive any information about the structure of $X_{r,n}$ by applying Theorem 3.2.1;

(2) when r is odd but bigger than 3, a similar strategy will give some information about the middle Hodge numbers for $X_{r,n}$, but we will still have unknown Hodge numbers in the middle;

(3) Theorem 3.2.1 excludes $n = 8$ case, while Theorem 3.1.1 includes $X_{3,8}$.

In the next chapter, we will introduce our standard computational machinery. In Section 4.2, we deal with examples as in (1); we give an alternative method other than the proof in Section 3.3 to compute the unknown Hodge numbers for (2) in Section 4.3; we also solve the Hodge diamond of $X_{3,8}$, which is out of the scope of Corollary 3.1.

3.3 Proof of main theorem

Proof. Note that $\dim X = 2(2m-1) - 3 = 4m - 5$. We have showed by the Lefschetz

Hyperplane theorem:

$$h^{p,q} = 0, \text{for } p + q < 4m - 5, p \neq q;$$

$$h^{p,p} = [\frac{p+1}{2}], \text{for } 2p < 4m - 5.$$

By Poincáre duality (or symmetry of Hodge diamond), we know that

$$h^{p,q} = 0, \text{for } p + q > 4m - 5, p \neq q;$$

$$h^{p,p} = h^{4m-5-p,4m-5-p} = [\frac{4m-5-p+1}{2}], \text{for } 2p > 4m - 5.$$

The above shows the pattern of middle column of Hodge diamond, namely,

$$1, 1, 2, 2, \cdots, m-1, m-1, m-1, m-1, m-1, \cdots, 2, 2, 1, 1$$

Now we need to show that all Hodge numbers in the middle row (i.e. $h^{p,q}$ for $p + q = 4m - 5$) are 0. It suffices to show that the middle cohomology vanishes, i.e. $H^{4m-5}(X) = 0$. We do this by computing the Euler characteristic of X.

Let $X = Gr(2, 2m+1) \cap H_1 \cap H_2 \cap H_3$, where H_1, H_2, H_3 are three hyperplanes in the Plücker space in generic positions. Then H_1, H_2, H_3 produce a $\mathbb{P}^2$-family of hyperplanes; for every $\lambda = [a_1, a_2, a_3] \in \mathbb{P}^2$, let $H_\lambda = a_1 H_1 + a_2 H_2 + a_3 H_3$. Then we obtain an incidence variety $I \subset Gr(2, 2m+1) \times \mathbb{P}^2$, whose fiber over a point $\lambda = [a_1, a_2, a_3] \in \mathbb{P}^2$ is exactly $Gr(2, 2m+1) \cap H_\lambda$.

We denote the projection of I onto the two factor by p_1 and p_2 respectively. For every $\lambda \in \mathbb{P}^2, p_2^{-1} = Gr(2, 2m+1) \cap H_\lambda$, which is a hyperplane section of $Gr(2, 2m+1)$. By Bertini's Theorem, for a generic λ, $p_2^{-1}(\lambda)$ is smooth, whose Euler characteristic $\chi(p_2^{-1}(\lambda)) = 2m^2$(Proposition 2.3 in [7]). In fact, this is true for all $\lambda \in \mathbb{P}^2$, because it is known that: if we parametrize all hyperplane sections of $Gr(2, V)$ where $V = \mathbb{C}^{2m+1}$ by $\mathbb{P}(\wedge^2 V^*)$, then the locus which parametrizes the singular hyperplane sections has codimension 3 in $\mathbb{P}(\wedge^2 V^*)$. Therefore a generic $\mathbb{P}^2$-family of hyperplanes misses the singular locus.

So

$$\chi(I) = \chi(p_2^{-1}(\lambda)) \cdot \chi(\mathbb{P}^2) = 2m^2 \cdot 3 = 6m^2 \tag{3.1}$$

Now we look at the other projection p_1. For any point $q \in Gr(2, 2m+1)$, we look at $p_1^{-1}(q)$. Since q also represents a point in the Plücker space $p_1^{-1}(q)$ is exactly the pencil of hyperplanes in our $\mathbb{P}^2$-family containing the point q, for a generic $q \in Gr(2, 2m+1)$. The only exception is that, when q lies in the base locus of the $\mathbb{P}^2$-family, namely $q \in X, p_1^{-1}(q) = \mathbb{P}^2$. In other words

$$p_1^{-1}(q) = \begin{cases} \mathbb{P}^1 & \text{if } q \in Gr(2, 2m+1) \setminus X \\ \mathbb{P}^2 & \text{if } q \in X \end{cases}$$

Therefore,

$$\begin{aligned}\chi(I) &= \chi(\mathbb{P}^1)\cdot\chi(Gr(2,2m+1)\setminus X) + \chi(\mathbb{P}^2)\cdot\chi(X)\\ &= 2(\chi(Gr(2,2m+1)) - \chi(X)) + 3\chi(X)\\ &= 2\chi(Gr(2,2m+1)) + \chi(X)\\ &= 2\begin{pmatrix} 2m+1 \\ 2 \end{pmatrix} + \chi(X).\end{aligned}$$

Combine the above equation and (3.1):

$$6m^2 = 2\begin{pmatrix} 2m+1 \\ 2 \end{pmatrix} + \chi(X);$$

this gives that

$$\chi(X) = 6m^2 = (2m+1)\cdot 2m = 2m^2 - 2m.$$

At the beginning of the proof we have figured out all other Hodge numbers. In particular, all Betti numbers except the middle one are known. They are as follows:

$b_{odd} = 0$ except the middle one b_{4m-5};

b_{even} attains values in the order of $(1,1,2,2,3,3,\cdots,m-2,m-2,m-1,m-1,m-1,m-1,m-2,m-2,\cdots,2,2,1,1)$.

Since the alternating sum equals $\chi(X)$, we have

$$\begin{aligned}\chi(X) &= b_0 - b_1 + b_2 - b_3 + \cdots \\ &= (1+1+2+2+\cdots+(m-1)+(m-1)) \times 2 - b_{4m-5} \\ &= m(m-1) \times 2 - b_{4m-5} \\ &= 2m^2 - 2m - b_{4m-5}\end{aligned}$$

where implies $b_{4m-5} = 0$, hence all Hodge numbers in the middle row $h^{p,q} = 0$ for any $p+q = 4m-5$. That leaves the Hodge structure of $X_{3,2m+1}$ of pure type.

For the even case $X_{3,2m}$, similar procedure suggests that the middle Betti number

$$b_{4m-7} = m^2 - 3m + 2,$$

which is also given in the proof of lemma 2.6 in [7].

Combining the Betti number with Corollary 3.1 and the result that $X_{3,8}$ has the following Hodge diamond:

$$
\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1 \\
&&&&&&&&0&&0 \\
&&&&&&&0&&1&&0 \\
&&&&&&0&&0&&0&&0 \\
&&&&&0&&0&&2&&0&&0 \\
&&&&0&&0&&0&&0&&0&&0 \\
&&&0&&0&&0&&2&&0&&0&&0 \\
&&0&&0&&0&&0&&0&&0&&0&&0 \\
&0&&0&&0&&0&&3&&0&&0&&0&&0 \\
0&&0&&0&&0&&3&&3&&0&&0&&0&&0 \\
&0&&0&&0&&0&&3&&0&&0&&0&&0 \\
&&0&&0&&0&&0&&0&&0&&0&&0 \\
&&&0&&0&&0&&2&&0&&0&&0 \\
&&&&0&&0&&0&&0&&0&&0 \\
&&&&&0&&0&&2&&0&&0 \\
&&&&&&0&&0&&0&&0 \\
&&&&&&&0&&1&&0 \\
&&&&&&&&0&&0 \\
&&&&&&&&&1
\end{array}
$$

The concrete computation refers to Section 4.3 of next chapter. We conclude that the Hodge pattern for $X_{3,2m}$ is as shown in Theorem 3.1.1 (b).

$$
\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}
$$

The concrete computation refers to Section 4.3 of next chapter. We conclude that the Hodge pattern for $X_{3,2m}$ is as shown in Theorem 3.1.1 (b).

of G. Further, it is easy to show that any two maximal tori of G are conjugate to each other. We fix a particular maximal torus T.

A root system is a key notion in the representation theory of semi-simple Lie algebras. Let E be a Euclidean vector space with inner product $(\cdot, \cdot)$. A *root system* is a finite spanning set $R \subset E$ such that for every $u \in R$ the orthogonal reflection

$$v \longrightarrow v - 2\frac{(u,u)}{(u,v)}u, v \in E$$

preserves R. A subset $R^+ \subset R$ is called a set of *positive roots* if there is a vector $v \in E$ such that $(\alpha, v) > 0$ if $\alpha \in R^+$ and $(\alpha, v) < 0$ if $\alpha \in R \setminus R^+$. A root is called *simple* if it is positive, and not the sum of any two positive roots. The simple roots form a basis of the vector space E , and any positive root is a positive integer linear combination of simple roots.

Recall that in the context of representations of algebraic and Lie groups, a weight of a representation is a generalization of the notion of an eigenvalue, and the corresponding eigenspace is called a weight space. In particular, a *weight of the representation V* is a weight λ such that the corresponding weight space is nonzero. Nonzero elements of the weight space are called *weight vectors.* For a representation of G on V, $v \in V$ is called a highest weight vector if it is annihilated by the action of all positive root spaces. If the highest weight vector v is in the weight space V_λ, then λ is the *highest weight* of the representation.

G is a complex semi-simple Lie group then any maximal solvable subgroup B of G is called *a Borel subgroup.* All Borel subgroups of a given group are conjugate. In the following, we shall fix a maximal torus T along with a Borel subgroup B which contains T. For λ an integral weight of T, let L_λ denote the corresponding equivariant line bundle over G/B. Indeed, λ defines in a natural way a one dimensional representation C_λ of B, by pulling back the representation on $T = B/U$, where U is the unipotent radical of B, namely, $B = T \ltimes U$. In other words, we can think of the projection map $G \to G/B$ as a principal B-bundle. Identifying L_λ with its sheaf of holomorphic sections, we consider its sheaf cohomology groups. Since G acts on the total space of the bundle L_λ by bundle automorphisms, this action naturally gives a G-module structure on these groups. The Borel-Weil-Bott theorem gives an explicit description of these groups as G-modules.

Before we state the theorem, we first need to describe the Weyl group action centered at ρ, one half of the sum of positive roots of G. For integral weight λ and w in the Weyl group W, we set

$$w * \lambda = w(\lambda + \rho) - \rho.$$

It is straightforward to check that this defines a group action, although this action is not linear, unlike the usual Weyl group action. Also, we say a weight λ is *dominant* if $\lambda(\alpha^\vee) \geq 0$ for all simple roots α, where the coroot $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

Weyl groups have a Bruhat order and length function ℓ in terms of this presentation: the length of a Weyl group element is the length of the shortest word representing that element in terms of these standard generators, which are the reflections given by simple roots. For example, the root system of A_2 consists of the vertices of a regular hexagon centered at the origin. The Weyl group of this root system is a subgroup of index two of the dihedral group of order 12. It is isomorphic to S_3, the symmetric group generated by the three reflections on the main diagonals of the hexagon.

Theorem 4.1.1. *(Borel-Weil-Bott[18]) For ℓ a length function on W, there are two possible cases:*

(1) There is no $w \in W$ such that $w * \lambda$ is dominant, in which case $H^i(G/B, L_\lambda) = 0$ for all i.

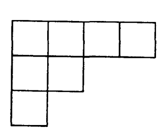
(2) There is a unique $w \in W$ such that $w*\lambda$ is dominant, in which case $H^i(G/B, L_\lambda) = 0$ for all $i \neq \ell(w)$ and $H^{\ell(w)}(G/B, L_\lambda)$ is the dual of the irreducible highest-weight representation of G with highest weight λ.

4.1.2 Formula for $\chi(\wedge^k T(m))$ on $Gr(2, n)$

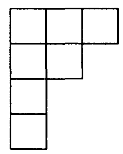
In below all theorems used refer to the book "Cohomology of Vector Bundles and Syzygies" by J. Weyman [19].

A partition λ of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ such that $n = \sum_{i=1}^{s} \lambda_i$. We also define $|\lambda| = n$. Typically it is required that $\lambda_s > 0$, but it is convenient to not make this restriction. Let i be the largest integer such that $\lambda_i > 0$; we say λ is a *partition* of n into i parts.

Given our partition we draw a picture of boxes, called the *Young diagram.* We put λ_i boxes in the i^{th} row. For example, for the partition $(4, 2, 1)$, we draw the following picture:



The Young diagram of the conjugate (or dual) partition λ' of λ is obtained from the Young diagram of λ by reflecting in the line $y = -x$. For example, when $\lambda = (4, 2, 1)$, λ' is $(3, 2, 1, 1)$ and the Young diagram is



In order to compute $\chi(\wedge^k T(m))$ when the underlying manifold is $Gr(2, n)$ for small k, we first introduce the Schur modules and Weyl modules.

Let us fix a free module E of dimension n over a commutative ring K. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of a number n. We consider the module

$$L_\lambda E = \bigwedge^{\lambda_1} E \otimes \bigwedge^{\lambda_2} E \otimes \cdots \otimes \bigwedge^{\lambda_s} E$$

and the submodule $R(\lambda, E)$, defined as the sum of submodules:

$$\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_{a-1}} E \otimes R_{a,a+1}(E) \otimes \bigwedge^{\lambda_{a+2}} E \otimes \cdots \otimes \bigwedge^{\lambda_s} E$$

for $1 \leq a \leq s - 1$, where $R_{a,a+1}(E)$ is the submodule spanned by the images of the following maps $\theta(\lambda, a, u, v; E)$ (more detailed definition can be found on Page 40 of

[19]) with $u + v < \lambda_{a+1}$:

$$\begin{array}{c}
\bigwedge^{u} E \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} E \otimes \bigwedge^{v} E \\
\downarrow \\
\bigwedge^{u} E \otimes \bigwedge^{\lambda_a - u} E \otimes \bigwedge^{\lambda_{a+1} - v} E \otimes \bigwedge^{v} E \\
\downarrow \\
\bigwedge^{\lambda_a} E \otimes \bigwedge^{\lambda_{a+1}} E
\end{array}$$

The space $L_\lambda E$ is called *the Schur module corresponding to the partition* λ. To define *the Weyl module* $K_\lambda E$ we take the definition of $L_\lambda E$, but instead of exterior powers we use divided powers, and instead of symmetric powers we use exterior powers. When the underlying field is the complex numbers, $K_\lambda E = L_{\lambda'} E$ where λ' is the dual partition of λ.

We have seen this tautological sequence

$$0 \longrightarrow R \longrightarrow \mathcal{O}_{Gr(2,n)}^{\oplus n} \longrightarrow Q \longrightarrow 0,$$

where R is the rank 2 tautological subbundle and Q is the rank $n-2$ tautological quotient bundle.

First we write the bundle $\wedge^k T(m)$ in terms of Weyl modules:

$$\begin{aligned}\wedge^k T(m) &= \wedge^k T \otimes \mathcal{O}(m)\\ &= \wedge^k (R^\vee \otimes Q) \otimes (\wedge^2 R^\vee)^{\otimes m}\end{aligned}$$

By corollary 2.3.3(b) of [19],

$$\wedge^k (R^\vee \otimes Q) = \oplus_{|\lambda|=m} K_\lambda R^\vee \otimes K_{\lambda'} Q;$$

as we mentioned, λ' is the dual partition of λ and we have the functorial isomorphisms $K_\lambda R^\vee = (L_{\lambda'} R)^\vee$.

By plugging in small k, we get the following formulas:

$$\begin{aligned}
k=1, &\qquad \wedge^1 (R^\vee \otimes Q) = R^\vee \otimes Q = K_{\ydiagram{1}} R^\vee \otimes K_{\ydiagram{1}} Q\\
k=2, &\qquad \wedge^2 (R^\vee \otimes Q) = (K_{\ydiagram{1,1}} R^\vee \otimes K_{\ydiagram{2}} Q) \oplus (K_{\ydiagram{2}} R^\vee \otimes K_{\ydiagram{1,1}} Q)\\
k=3, &\qquad \wedge^3 (R^\vee \otimes Q) = (K_{\ydiagram{2,1}} R^\vee \otimes K_{\ydiagram{2,1}} Q) \oplus (K_{\ydiagram{3}} R^\vee \otimes K_{\ydiagram{1,1,1}} Q)
\end{aligned}$$

Note that the term $K_{\ydiagram{1,1,1}} R^\vee \otimes K_{\ydiagram{3}} Q$ does not appear, since rank $R^\vee = 2$. Furthermore the term $K_{\ydiagram{3}} R^\vee \otimes K_{\ydiagram{1,1,1}} Q$ appears only when rank $Q > 2$, i.e. $n \geq 5$.

$$k=4, \qquad \wedge^4(R^\vee \otimes Q) = (K_{\ydiagram{4}}R^\vee \otimes K_{\ydiagram{1,1,1,1}}Q) \oplus (K_{\ydiagram{3,1}}R^\vee \otimes K_{\ydiagram{2,1,1}}Q) \oplus (K_{\ydiagram{2,2}}R^\vee \otimes K_{\ydiagram{2,2}}Q)$$

For the same reason, the first term above only appears when $n \geq 6$.

By Example 2.1.17(b), $\wedge^2 R^\vee = K_{\ydiagram{1,1}}R^\vee$.

By Theorem 2.3.4, $(\wedge^2 R^\vee)^{\otimes m} = (K_{\ydiagram{1,1}}R^\vee)^{\otimes m} = K_{\underbrace{\ydiagram{6,6}}_{m \text{ columns}}}R^\vee = K_{(m,m)}R^\vee$.

Again by Theorem 2.3.4, we get the formula for $\wedge^k T(m)$:

when $k=1$,

$$\begin{aligned} T(m) &= K_{(1)}R^\vee \otimes K_{(1)}Q \otimes K_{(m,m)}R^\vee \\ &= K_{(m+1,m)}R^\vee \otimes K_{(1)}Q \\ &= K_{(m+1,m)}R^\vee \otimes K_{(\underbrace{1,\ldots,1}_{(n-3)\ 1\text{'s}},0)}Q^\vee; \end{aligned}$$

when $k=2$,

$$\begin{aligned} \wedge^2 T(m) &= ((K_{(2)}R^\vee \otimes K_{(1,1)}Q) \oplus (K_{(1,1)}R^\vee \otimes K_{(2)}Q)) \otimes K_{(m,m)}R^\vee \\ &= (K_{(m+2,m)}R^\vee \otimes K_{(1,1)}Q) \oplus (K_{(m+1,m+1)}R^\vee \otimes K_{(2)}Q) \\ &= K_{(m+2,m)}R^\vee \otimes K_{(\underbrace{1,\ldots,1}_{(m-4)\ 1\text{'s}},0,0)}Q^\vee \oplus K_{(m+1,m+1)}R^\vee \otimes K_{(\underbrace{2,\ldots,2}_{(m-3)\ 2\text{'s}},0)}Q^\vee; \end{aligned}$$

when $k=3$,

$$\begin{aligned}\wedge^3 T(m) &= ((K_{(3)}R^\vee \otimes K_{(1,1,1)}Q) \oplus (K_{(2,1)}R^\vee \otimes K_{(2,1)}Q)) \otimes K_{(m,m)}R^\vee \\ &= (K_{(m+3,m)}R^\vee \otimes K_{(1,1,1)}Q) \oplus (K_{(m+2,m+1)}R^\vee \otimes K_{(2,1)}Q) \\ &= K_{(m+3,m)}R^\vee \otimes K_{(\underbrace{1,\ldots,1}_{(m-5)\text{ 1's}},0,0,0)}Q^\vee \oplus K_{(m+2,m+1)}R^\vee \otimes K_{(\underbrace{2,\ldots,2}_{(m-4)\text{ 2's}},1,0)}Q^\vee;\end{aligned}$$

when $k=4$,

$$\begin{aligned}\wedge^4 T(m) &= ((K_{(4)}R^\vee \otimes K_{(1,1,1,1)}Q) \oplus (K_{(3,1)}R^\vee \otimes K_{(2,1,1)}Q) \\ &\oplus (K_{(2,2)}R^\vee \otimes K_{(2,2)}Q)) \otimes K_{(m,m)}R^\vee \\ &= (K_{(m+4,m)}R^\vee \otimes K_{(1,1,1,1)}Q) \oplus (K_{(m+3,m+1)}R^\vee \otimes K_{(2,1,1)}Q) \\ &\oplus (K_{(m+2,m+2)}R^\vee \otimes K_{(2,2)}Q) \\ &= K_{(m+4,m)}R^\vee \otimes K_{(\underbrace{1,\ldots,1}_{(m-6)\text{ 1's}},0,0,0,0)}Q^\vee \oplus K_{(m+3,m+1)}R^\vee \otimes K_{(\underbrace{2,\ldots,2}_{(m-5)\text{ 2's}},1,1,0)}Q^\vee \\ &\oplus K_{(m+2,m+2)}R^\vee \otimes K_{(\underbrace{2,\ldots,2}_{(m-4)\text{ 2's}},0,0)}Q^\vee.\end{aligned}$$

Note that in all our computations, part (2) of Corollary 4.1.9 of [19] applies. The unique permutation we take is $\sigma = id$ and therefore $l(\sigma) = 0$.

$$k = 1, \qquad \wedge^k T(m) = K_{(m+2,m+1,\underbrace{1,\,\ldots,\,1}_{(m-3)\ 1\text{'s}},0)}\mathbb{C}^n$$

$$k = 2, \qquad \wedge^k T(m) = K_{(m+3,m+1,\underbrace{1,\,\ldots,\,1}_{(m-4)\ 1\text{'s}},0,0)}\mathbb{C}^n \oplus K_{(m+3,m+3,\underbrace{2,\,\ldots,\,2}_{(m-3)\ 2\text{'s}},0)}\mathbb{C}^n$$

$$k = 3, \qquad \wedge^k T(m) = K_{(m+4,m+1,\underbrace{1,\,\ldots,\,1}_{(m-5)\ 1\text{'s}},0,0,0)}\mathbb{C}^n \oplus K_{(m+4,m+3,\underbrace{2,\,\ldots,\,2}_{(m-4)\ 2\text{'s}},1,0)}\mathbb{C}^n$$

$$k = 4, \qquad \wedge^k T(m) = K_{(m+5,m+1,\underbrace{1,\,\ldots,\,1}_{(m-6)\ 1\text{'s}},0,0,0,0)}\mathbb{C}^n \oplus K_{(m+5,m+3,\underbrace{2,\,,\,..,\,2}_{(m-5)\ 2\text{'s}},1,1,0)}\mathbb{C}^n$$

$$\oplus K_{(m+4,m+4,\underbrace{2,\,\ldots,\,2}_{(m-4)\ 2\text{'s}},0,0)}\mathbb{C}^n$$

Finally, by Theorem 2.2.10, we know the subscripts above are exactly the highest weights of the corresponding representations.

Combining all above, we get the following formulas:

$$
\begin{aligned}
\chi(T(m)) &= \dim K_{(m+2,m+1,1,\cdots,1,0)}\mathbb{C}^n \\
\chi(\wedge^2 T(m)) &= \dim K_{(m+3,m+1,1,\cdots,1,0,0)}\mathbb{C}^n \\
&\quad + \dim K_{(m+3,m+3,2,\cdots,2,0)}\mathbb{C}^n \\
\chi(\wedge^3 T(m)) &= \dim K_{(m+4,m+1,1,\cdots,1,0,0,0)}\mathbb{C}^n \\
&\quad + \dim K_{(m+4,m+3,2,\cdots,2,1,0)}\mathbb{C}^n, \qquad \text{for } n \geq 5 \\
\chi(\wedge^4 T(m)) &= \dim K_{(m+5,m+1,1,\cdots,1,0,0,0,0)}\mathbb{C}^n \\
&\quad + \dim K_{(m+5,m+3,2,\cdots,2,1,1,0)}\mathbb{C}^n \\
&\quad + \dim K_{(m+4,m+4,2,\cdots,2,0,0)}\mathbb{C}^n, \qquad \text{for } n \geq 6.
\end{aligned}
$$

The last thing we need in the following computation is a special case of Weyl's character formula [17]: for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0)$, we have

$$
\dim K_\lambda \mathbb{C}^n = \prod_{1 \leq i < j \leq s} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
$$

4.2 Examples in codimension $r \geq 4$

4.2.1 $X_{6,6}$: the codimension 6 smooth linear section in $Gr(2,6)$

In section 2.2, we showed that $X_{6,6}$ is a K3 surface of genus 8. Therefore, the Hodge diamond of $X_{6,6}$ is known:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We will apply our machinery to compute the Hodge numbers of $X_{6,6}$. We understand it may be overkill for this particular problem, but we indeed use this as a guiding example for the general method we will introduce.

We recall the Koszul complex, a major tool we use in our computations. Let $f : P \longrightarrow Q$ be a A-module homomorphism between projective A-modules. The *n-th Koszul complex* associated with f is denoted by $Kos^n(f)$ and defined as follows:

$$Kos^n(f)_k = \wedge^k(P) \otimes Sym^{n-k}(Q)$$

and the Koszul differential $d_{k+1} : Kos^n(f)_{k+1} \longrightarrow Kos^n(f)_k$ is given by

$$p_1 \wedge \cdots \wedge p_{k+1} \otimes q_{k+2} \cdots q_n \to \sum_{i=1}^{k+1} (-1)^{k+1-i} p_1 \wedge \cdots \wedge \check{p}_i \wedge \cdots \wedge p_{k+1} \otimes f(p_i) q_{k+2} \cdots q_n.$$

Since X is the codimension 6 smooth linear intersection in $Gr(2,6)$, we have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,6)} \mid_X \longrightarrow \Omega^1_X \longrightarrow 0. \tag{4.1}$$

The regular sequence determines the Koszul resolution associated with the restriction map $f : \mathcal{O}_{Gr(2,6)} \to \mathcal{O}_X$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{Gr(2,6)}(-6) \longrightarrow \mathcal{O}_{Gr(2,6)}(-5)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,6)}(-4)^{\oplus 15} \longrightarrow \mathcal{O}_{Gr(2,6)}(-3)^{\oplus 20} \\ \longrightarrow \mathcal{O}_{Gr(2,6)}(-2)^{\oplus 15} \longrightarrow \mathcal{O}_{Gr(2,6)}(-1)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,6)} \longrightarrow \mathcal{O}_X \longrightarrow 0. \end{aligned} \tag{4.2}$$

To obtain $\chi(X)$, it suffices to compute $\chi(\mathcal{O}_{Gr(2,6)}(m))$.

Without loss of generality, we only need to determine $\chi(\mathcal{O}_{Gr(2,6)}(m))$ for positive m, because what we will get is a polynomial in terms of m.

Borel-Weil-Bott Theorem tells us that all the higher cohomology of $\mathcal{O}_{Gr(2,6)}(m)$ vanishes and also

$$\chi(\mathcal{O}_{Gr(2,6)}(m)) = \dim(\Gamma(\mathcal{O}_{Gr(2,6)}(m))) = \dim K_{m\lambda}V,$$

where $V = \wedge^2$(standard representation of SL_6), and hence $\lambda = (1,1,0,0,0,0)$.

We check if the weight λ works by applying Weyl's dimension formula.

$$\begin{aligned} \dim K_{(1,1,0,0,0,0)}V &= \prod_{1\le i<j\le 6} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\ &= \prod_{j=3}^{6} \frac{1-0+j-1}{j-1} \cdot \prod_{j=3}^{6} \frac{1-0+j-2}{j-2} \\ &= \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4} \\ &= 15 \\ &= \dim(\Gamma(\mathcal{O}_{Gr(2,6)}(1))), \end{aligned}$$

which agrees with the number of Plücker relations.

Similarly,

$$\begin{aligned}
\dim K_{(2,2,0,0,0,0)}V &= \prod_{1\le i<j\le 6} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\
&= \prod_{j=3}^{6} \frac{j-1+2}{j-1} \cdot \prod_{j=3}^{6} \frac{j}{j-2} \\
&= \frac{4}{2}\cdot\frac{5}{3}\cdot\frac{6}{4}\cdot\frac{7}{5}\cdot\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}\cdot\frac{6}{4} \\
&= 105 \\
&= \dim(\Gamma(\mathcal{O}_{Gr(2,6)}(2))).
\end{aligned}$$

Note this also agrees with the number of quadric in $\mathbb{P}^{14}$ minus the number of Plücker relations $= \begin{pmatrix} 14+2 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 120 - 15 = 105$, where 120 is the number of quadrics in $\mathbb{P}^{14}$ and 15 is the number of Plücker relations.

In general, we can get

$$\chi(\mathcal{O}_{Gr(2,6)}(m)) = \dim(\Gamma(\mathcal{O}_{Gr(2,6)}(m))) = \prod_{j=3}^{6} \frac{j-1+m}{j-1} \cdot \prod_{j=3}^{6} \frac{j-2+m}{j-2}.$$

Plugging this back to (4.2), we find

$$\begin{aligned}\chi(\mathcal{O}_X) &= \chi(\mathcal{O}_{Gr(2,6)}) - 6\cdot\chi(\mathcal{O}_{Gr(2,6)}(-1)) + 15\cdot\chi(\mathcal{O}_{Gr(2,6)}(-2)) - 20\cdot\chi(\mathcal{O}_{Gr(2,6)}(-3))\\ &\quad + 15\cdot\chi(\mathcal{O}_{Gr(2,6)}(-4)) - 6\cdot\chi(\mathcal{O}_{Gr(2,6)}(-5)) + \chi(\mathcal{O}_{Gr(2,6)}(-6)))\\ &= 1-0+0-0+0-0+1\\ &= 2.\end{aligned}$$

To compute $\chi(\Omega^1_X)$

Next we want to compute $\chi(\Omega^1_X)$. Tensoring the exact sequence (4.2) with $\Omega^1_{Gr(2,6)}$ gives

$$\begin{aligned}0 &\longrightarrow \Omega^1_{Gr(2,6)}(-6) \longrightarrow \Omega^1_{Gr(2,6)}(-5)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,6)}(-4)^{\oplus 15} \longrightarrow \Omega^1_{Gr(2,6)}(-3)^{\oplus 20}\\ &\longrightarrow \Omega^1_{Gr(2,6)}(-2)^{\oplus 15} \longrightarrow \Omega^1_{Gr(2,6)}(-1)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,6)}|_X \longrightarrow \Omega^1_X \longrightarrow 0.\end{aligned} \tag{4.3}$$

By Serre duality, $H^i(\Omega^1_{Gr(2,6)}) \cong H^{n-i}(T_{Gr(2,6)}\otimes\omega_{Gr(2,6)})^\vee$, which gives $\chi(\Omega^1_{Gr(2,6)}(m)) = \chi(T_{Gr(2,6)}\otimes\omega_{Gr(2,6)}(-m)) = \chi(T_{Gr(2,6)}(-6-m))$, thus the question is to identify appropriate representations for $T_{Gr(2,6)}$.

Recall the formula at the end of Section 4.1:

$$\dim(\Gamma(T_{Gr(2,6)}(m))) = \dim(K_{(m+2,m+1,1,1,1,0)}V).$$

Then the Weyl character formula tells us that

$$\dim K_{(m+2,m+1,1,1,1,0)}V = \frac{(m+1)(m+2)(m+3)^2(m+4)(m+5)^2(m+7)}{3\cdot 5!}.$$

Let's look back to the long exact sequence (4.3),

$$\begin{aligned}
\chi(\Omega^1_{Gr(2,6)}|_X) &= \chi(\Omega^1_{Gr(2,6)}) - 6\chi(\Omega^1_{Gr(2,6)}(-1)) + \cdots \\
&= \chi(T_{Gr(2,6)}(-6)) - 6\chi(T_{Gr(2,6)}(-5)) + 15\chi(T_{Gr(2,6)}(-4)) - 20\chi(T_{Gr(2,6)}(-3)) \\
&+ 15\chi(T_{Gr(2,6)}(-2)) - 6\chi(T_{Gr(2,6)}(-1)) + \chi(T_{Gr(2,6)}) \\
&= -1 - 0 + 0 - 0 + 0 - 0 + 35 \\
&= 34.
\end{aligned}$$

Then from (4.1), we find that $\chi(\Omega^1_X) = \chi(\Omega^1_{Gr(2,6)}|_X) - 6\cdot\chi(\mathcal{O}_X(-1))$, the first term of the right side is 34 as we computed. To compute the second term, we need to use (4.2) again, shifted it by -1.

So (4.2) becomes

$$\begin{aligned}
0 &\longrightarrow \mathcal{O}_{Gr(2,6)}(-7) \longrightarrow \mathcal{O}_{Gr(2,6)}(-6)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,6)}(-5)^{\oplus 15} \longrightarrow \mathcal{O}_{Gr(2,6)}(-4)^{\oplus 20} \\
&\longrightarrow \mathcal{O}_{Gr(2,6)}(-3)^{\oplus 15} \longrightarrow \mathcal{O}_{Gr(2,6)}(-2)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,6)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0;
\end{aligned}$$

a similar alternating sum gives that $\chi(\mathcal{O}_X(-1)) = 9$, which is how we get

$$\chi(\Omega^1_X) = 34 - 6\cdot 9 = -20.$$

Thus we get the Hodge diamond for X:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

To compute $\chi(\Omega_X^2)$

We can keep using this method to compute $\chi(\Omega_X^2)$, which takes a little bit more effort.

We start with sequence (4.2), if we shift by -2 and then take the alternating sum of the long exact sequence, we yield the following equation:

$$\begin{aligned}\chi(\mathcal{O}_X(-2)) = \chi(\mathcal{O}_{Gr_{(2,6)}}(-2)) - 6\chi(\mathcal{O}_{Gr_{(2,6)}}(-3)) + 15\chi(\mathcal{O}_{Gr_{(2,6)}}(-4)) - 20\chi(\mathcal{O}_{Gr_{(2,6)}}(-5)) \\ + 15\chi(\mathcal{O}_{Gr_{(2,6)}}(-6)) - 6\chi(\mathcal{O}_{Gr_{(2,6)}}(-7)) + \chi(\mathcal{O}_{Gr_{(2,6)}}(-8)).\end{aligned}$$

We divide the computation process into three parts. Before the computation, we would like to make a notational remark: $\mathcal{O}(m)$ always means $\mathcal{O}_{Gr(2,n)}(m)$, $T(m)$ is just $T_{Gr(2,n)}(m)$ and $\Omega^i(m)$ is short for $\Omega^i_{Gr(2,n)}(m)$, where n depends on the specific example, for instance, $n = 6$ in this case.

Part I:

$$\begin{aligned}\chi(\mathcal{O}(m)) &= \dim K_{(m,m,0,0,0,0)}V\\ &= \frac{(m+2)(m+3)(m+4)(m+5)}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{(m+1)(m+2)(m+3)(m+4)}{1\cdot 2\cdot 3\cdot 4}\\ \chi(\mathcal{O}(-2)) &= \chi(\mathcal{O}(-3)) = \chi(\mathcal{O}(-4)) = \chi(\mathcal{O}(-5)) = 0\\ \chi(\mathcal{O}(-6)) &= \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4} = 1\\ \chi(\mathcal{O}(-7)) &= \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4} = 15\\ \chi(\mathcal{O}(-8)) &= \frac{6\cdot 5\cdot 4\cdot 3}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4} = 105\end{aligned}$$

So $\chi(\mathcal{O}_X(-2)) = 15\cdot 1 - 6\cdot 15 + 105 = 30$.

Part II:

$$\begin{aligned}\chi(\Omega^1(-1)\big|_X) &= \chi(\Omega^1(-1)) - 6\chi(\Omega^1(-2)) + 15\chi(\Omega^1(-3)) - 20\chi(\Omega^1(-4))\\ &\quad + 15\chi(\Omega^1(-5)) - 6\chi(\Omega^1(-6)) + \chi(\Omega^1(-7))\\ &= \chi(T(-5)) - 6\chi(T(-4)) + 15\chi(T(-3)) - 20\chi(T(-2))\\ &\quad + 15\chi(T(-1)) - 6\chi(T) + \chi(T(1))\end{aligned}$$

$$\begin{aligned}\chi(T(m)) &= \dim K_{(m+2,m+1,1,1,1,0)}V\\ &= \frac{2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+7}{5}\\ &\quad \frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot\frac{m+5}{4}\cdot\frac{4}{1}\end{aligned}$$

$$\chi(T(-5)) = \chi(T(-4)) = \chi(T(-3)) = \chi(T(-2)) = \chi(T(-1)) = 0$$

$$\chi(T) = \frac{2}{1}\cdot\frac{3\cdot4\cdot5\cdot7}{2\cdot3\cdot4\cdot5}\cdot\frac{1\cdot2\cdot3\cdot5}{1\cdot2\cdot3\cdot4}\cdot\frac{4}{1} = 35$$

$$\chi(T(1)) = \frac{2}{1}\cdot\frac{4\cdot5\cdot6\cdot8}{2\cdot3\cdot4\cdot5}\cdot\frac{2\cdot3\cdot4\cdot6}{1\cdot2\cdot3\cdot4}\cdot\frac{4}{1} = 384$$

So $\chi(\Omega^1(-1)|_X) = -6\cdot 35 + 384 = 174$.

Part III:

$$\begin{aligned}\chi(\Omega^2|_X) = \chi(\Omega^2) &- 6\chi(\Omega^2(-1) + 15\chi(\Omega^2(-2)) - 20\chi(\Omega^2(-3)) \\ &+ 15\chi(\Omega^2(-4)) - 6\chi(\Omega^2(-5)) + \chi(\Omega^2(-6))\end{aligned} \tag{4.4}$$

By using the second formula in Section 4.1, we have

$$\chi(\Omega^2(m)) = \chi(\wedge^2 T(m)) = \dim K_{(m+3,m+1,1,1,0,0)}V + \dim K_{(m+3,m+3,2,2,2,0)}V.$$

We will compute the contribution of the two terms separately.

$$\begin{aligned}\dim K_{(m+3,m+1,1,1,0,0)}V = &\frac{3}{1}\cdot\frac{m+4}{2}\cdot\frac{m+5}{3}\cdot\frac{m+7}{4}\cdot\frac{m+8}{5} \\ &\frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{4}{3}\end{aligned}$$

Plugging in the right values of m, we have $\chi(\Omega^2) = \frac{3}{1}\cdot\frac{4\cdot5\cdot7\cdot8}{2\cdot3\cdot4\cdot5}\cdot\frac{1\cdot2\cdot4\cdot5}{1\cdot2\cdot3\cdot4}\cdot 6 = 280$;

$\chi(\Omega^2(-1)) = \chi(\Omega^2(-2)) = 0$;

$\chi(\Omega^2(-3)) = \frac{3}{1} \cdot \frac{1 \cdot 2 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{2 \cdot 1 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 6 = 1;$

$\chi(\Omega^2(-4)) = \chi(\Omega^2(-5)) = 0;$

and $\chi(\Omega^2(-6)) = \frac{3}{1} \cdot \frac{2 \cdot 1 \cdot 1 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5 \cdot 4 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 6 = 1.$

So the alternating sum in equation (4.4) shows the total contribution to $\chi(\Omega^2|_X)$ is $280 - 20 \cdot 1 + 1$, which is just 261.

The second term has a similar dimension formula:

$$\begin{aligned}\dim K_{(m+3,m+3,2,2,2,0)}V = & \frac{m+3}{2} \cdot \frac{m+4}{3} \cdot \frac{m+5}{4} \cdot \frac{m+8}{5} \\ & \frac{m+2}{1} \cdot \frac{m+3}{2} \cdot \frac{m+4}{3} \cdot \frac{m+7}{4} \cdot \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{3}\end{aligned}$$

$\chi(\Omega^2) \Rightarrow \frac{3 \cdot 4 \cdot 5 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{2 \cdot 3 \cdot 4 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 10 = 280$

$\chi(\Omega^2(-1)) \Rightarrow \frac{2 \cdot 3 \cdot 4 \cdot 7}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1 \cdot 2 \cdot 3 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 10 = 21$

$\chi(\Omega^2(-2)), \chi(\Omega^2(-3)), \chi(\Omega^2(-4)), \chi(\Omega^2(-5)) \Rightarrow 0$

$\chi(\Omega^2(-6)) \Rightarrow \frac{3 \cdot 2 \cdot 1 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 10 = 1$

Again, equation (4.4) leads to the total contribution to $\chi(\Omega^2|_X)$ is $280 - 6 \cdot 21 + 1$, which is just 155.

Therefore, adding these two parts together gives $\chi(\Omega^2|_X) = 261 + 155 = 416$.

Summarizing the above three parts, we find $\chi(\Omega^2_X) = 30 \cdot 21 - 174 \cdot 6 + 416 = 2$, which is exactly as desired.

It is easy to check $X_{6,6}$ is a K3 surface, thus we know all the cohomology classes of right away. This is not always the case. Now let's move to the second example where we need to work to compute the Hodge diamond.

4.2.2 $X_{4,n}$: codimension 4 smooth linear sections in $Gr(2,n)$ with $n = 5,6,7$

$\mathbf{X_{4,7}}$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 15 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 2 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}
$$

Among these three with respect to $n = 5, 6, 7$, $X_{4,7}$ is of the highest dimension. It takes more work to complete the Hodge diamond of $X_{4,7}$, so we start with this one. As shown above, up to H^5, the Hodge numbers of $X_{4,7}$ are known. The reason for this is the Lefschetz hyperplane Theorem. And $H^0(X, \Omega^6_X) = H^0(X, K_X) = H^0(X, \mathcal{O}(-3))$, therefore $h^{6,0} = 0 = h^{0,6}$; the second equation follows from duality. We have the upper half of the Hodge diamond of $X_{4,7}$ and there are only three unknown Hodge numbers $h^{1,5} = h^{5,1}$, $h^{2,4} = h^{4,2}$ and $h^{3,3}$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 2 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & h^{1,5} & & h^{2,4} & & h^{3,3} & & h^{4,2} & & h^{5,1} & & 0
\end{array}
$$

To compute $\chi(\Omega^1_X)$

We start with the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,7)}\mid_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

We want $\chi(\Omega^1_X)$, which determines $h^j(\Omega^1_X)$ via Lefschetz.

The Koszul resolution for a regular sequence gives

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{Gr(2,7)}(-4) \longrightarrow \mathcal{O}_{Gr(2,7)}(-3)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,7)}(-2)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,7)}(-1)^{\oplus 4} \\ \longrightarrow \mathcal{O}_{Gr(2,7)} \longrightarrow \mathcal{O}_X \longrightarrow 0. \end{aligned} \tag{4.5}$$

Twisting the Koszul complex (4.5) by (-1), we have the following sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{Gr(2,7)}(-5) \longrightarrow \mathcal{O}_{Gr(2,7)}(-4)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,7)}(-3)^{\oplus 6} \\ \longrightarrow \mathcal{O}_{Gr(2,7)}(-2)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,7)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0. \end{aligned}$$

The Borel-Weil-Bott Theorem gives

$$\chi(\mathcal{O}_{Gr(2,7)}(m)) = \dim(\Gamma(\mathcal{O}_{Gr(2,7)}(m))) = \dim K_{m\lambda}V,$$

where $\lambda = \wedge^2$(standard representation of SL_7).

Since $\lambda = (1,1,0,0,0,0,0)$, we obtain

$$\begin{aligned}\chi(\mathcal{O}_{Gr(2,7)}(m)) &= \dim K_{(m,m,0,0,0,0,0)}V \\ &= \prod_{j=3}^{7} \frac{m+j-1}{j-1} \cdot \frac{m+j-2}{j-2} \\ &= \prod_{j=2}^{6} \frac{m+j}{j} \prod_{j=1}^{5} \frac{m+j}{j}\end{aligned}$$

by applying Weyl character formula.

Thus we have

$$\begin{aligned}\chi(\mathcal{O}_X(-1)) = \chi(\mathcal{O}_{Gr(2,7)}(-5)) - 4\chi(\mathcal{O}_{Gr(2,7)}(-4)) + 6\chi(\mathcal{O}_{Gr(2,7)}(-3)) \\ - 4\chi(\mathcal{O}_{Gr(2,7)}(-2)) + \chi(\mathcal{O}_{Gr(2,7)}(-1))\end{aligned}$$

and $m = -1,-2,-3,-4,-5$ all end up with $\chi(\mathcal{O}_{Gr(2,7)}(m)) = 0$, so it is straightforward to see that $\chi(\mathcal{O}_X(-1)) = 0$.

We tensor the exact sequence (4.5) with $\Omega^1_{Gr(2,7)}$ to get

$0 \longrightarrow \Omega^1_{Gr(2,7)}(-4) \longrightarrow \Omega^1_{Gr(2,7)}(-3)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,7)}(-2)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,7)}(-1)^{\oplus 4}$

$\longrightarrow \Omega^1_{Gr(2,7)} \longrightarrow \Omega^1_{Gr(2,7)}|_X \longrightarrow 0.$

However,

$$\chi(\Omega^1_{Gr(2,7)}(-m)) = \chi(T_{Gr(2,7)}(-7-m))$$

and analogously to the previous example,

$$\chi(T_{Gr(2,7)}(n)) = \dim K_{(2+n,1+n,1,1,1,1,0)}V$$
$$= \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)(n+6)^2(n+8)}{12 \cdot 6!}.$$

We eventually end up with $\chi(\Omega^1_{Gr(2,7)}|_X) = -1$, because $\chi(\Omega^1_{Gr(2,7)}) = -1$ and all the other terms contribute 0 in Euler characteristics. Combining this result with $\chi(\mathcal{O}_X(-1)) = 0$, we find that $\chi(\Omega^1_X) = \chi(\Omega^1_{Gr(2,7)}|_X) - 4\chi(\mathcal{O}_X(-1)) = -1-4\cdot 0 = -1$.

As we know, $\chi(\Omega^1_X) = h^{1,0}-h^{1,1}+h^{1,2}-h^{1,3}+h^{1,4}-h^{1,5} = 0-1+0-0+0-h^{1,5} = -1$, which shows that $h^{1,5}$ has to be 0.

To compute $\chi(\Omega^2_X)$

We start with

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,7)}|_X \longrightarrow \Omega^1_X \longrightarrow 0,$$

take the second exterior power to get:

$$0 \longrightarrow \mathcal{O}_X(-2)^{\oplus 10} \longrightarrow \Omega^1_{Gr(2,7)}(-1)^{\oplus 4}|_X \longrightarrow \Omega^2_{Gr(2,7)}|_X \longrightarrow \Omega^2_X \longrightarrow 0.$$

Thus we need to compute

$$\chi(\Omega^2_X) = 10\chi(\mathcal{O}_X(-2)) - 4\chi(\Omega^1_{Gr(2,7)}(-1)|_X) + \chi(\Omega^2_{Gr(2,7)}|_X).$$

Twisting by -2, (4.5) becomes

$$
\begin{aligned}
0 \longrightarrow \mathcal{O}_{Gr(2,7)}(-6) \longrightarrow \mathcal{O}_{Gr(2,7)}(-5)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,7)}(-4)^{\oplus 6} \\
\longrightarrow \mathcal{O}_{Gr(2,7)}(-3)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,7)}(-2) \longrightarrow \mathcal{O}_X(-2) \longrightarrow 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\chi(\mathcal{O}_X(-2)) = \chi(\mathcal{O}_{Gr(2,7)}(-6)) - 4\chi(\mathcal{O}_{Gr(2,7)}(-5)) + 6\chi(\mathcal{O}_{Gr(2,7)}(-4)) \\
- 4\chi(\mathcal{O}_{Gr(2,7)}(-3)) + \chi(\mathcal{O}_{Gr(2,7)}(-2))
\end{aligned}
$$

and

$$
\begin{aligned}
\chi(\mathcal{O}_{Gr(2,7)}(m)) &= \dim K_{(m,m,0,0,0,0,0)}V \\
&= \prod_{j=2}^{6} \frac{m+j}{j} \prod_{j=1}^{5} \frac{m+j}{j}.
\end{aligned}
$$

When $m = -2, -3, -4, -5, -6$, we all have $\chi(\mathcal{O}_{Gr(2,7)}(m)) = 0$. As a result, $\chi(\mathcal{O}_X(-2)) = 0$ as well.

Below, we move to work on $\chi(\Omega^1_{Gr(2,7)}(-1)|_X)$.

As above, by the Koszul resolution,

$$
\begin{aligned}
0 \longrightarrow \Omega^1_{Gr(2,7)}(-5) \longrightarrow \Omega^1_{Gr(2,7)}(-4)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,7)}(-3)^{\oplus 6} \\
\longrightarrow \Omega^1_{Gr(2,7)}(-2)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,7)}(-1) \longrightarrow \Omega^1_{Gr(2,7)}(-1)|_X \longrightarrow 0,
\end{aligned}
$$

we have

$$\chi(\Omega^1_{Gr(2,7)}(-1)|_X) = \chi(\Omega^1_{Gr(2,7)}(-5)) - 4\chi(\Omega^1_{Gr(2,7)}(-4)) + 6\chi(\Omega^1_{Gr(2,7)}(-3))$$
$$- 4\chi(\Omega^1_{Gr(2,7)}(-2)) + \chi(\Omega^1_{Gr(2,7)}(-1)).$$

Note that

$$\chi(\Omega^1_{Gr(2,7)}(n)) = \chi(T_{Gr(2,7)}(-n) \otimes \mathcal{O}_{Gr(2,7)}(-7)) = \chi(T_{Gr(2,7)}(-n-7)),$$

this gives

$$\chi(\Omega^1_{Gr(2,7)}(-1)|_X) = \chi(T_{Gr(2,7)}(-2)) - 4\chi(T_{Gr(2,7)}(-3)) + 6\chi(T_{Gr(2,7)}(-4))$$
$$- 4\chi(T_{Gr(2,7)}(-5)) + \chi(T_{Gr(2,7)}(-6)),$$

and

$$\begin{aligned}\chi(T_{Gr(2,7)}(m)) &= \dim K_{(m+2,m+1,1,1,1,1,0)}V \\ &= \frac{1}{2}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5}\cdot\frac{m+5}{6} \\ &\quad\cdot\frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot\frac{m+4}{4}\cdot\frac{m+6}{5} \\ &\quad\cdot\frac{5}{4}\cdot\frac{4}{3}\cdot\frac{3}{2}\cdot\frac{2}{1}.\end{aligned}$$

When $m = -2, -3, -4, -5, -6$, we have $\chi(T_{Gr(2,7)}(m)) = 0$. Thus, $\chi(\Omega^1_{Gr(2,7)}(-1)|_X) = 0$.

At this stage, we are ready to compute $\chi(\Omega^2_{Gr(2,7)}|x)$.

Again we use the Koszul complex

$$0 \longrightarrow \Omega^2_{Gr(2,7)}(-4) \longrightarrow \Omega^2_{Gr(2,7)}(-3)^{\oplus 4} \longrightarrow \Omega^2_{Gr(2,7)}(-2)^{\oplus 6}$$

$$\longrightarrow \Omega^2_{Gr(2,7)}(-1)^{\oplus 4} \longrightarrow \Omega^2_{Gr(2,7)} \longrightarrow \Omega^2_{Gr(2,7)}|x \longrightarrow 0$$

to get

$$\chi(\Omega^2_{Gr(2,7)}|x) = \chi(\Omega^2_{Gr(2,7)}(-4)) - 4\chi(\Omega^2_{Gr(2,7)}(-3)) + 6\chi(\Omega^2_{Gr(2,7)}(-2))$$

$$- 4\chi(\Omega^2_{Gr(2,7)}(-1)) + \chi(\Omega^2_{Gr(2,7)}).$$

Note that

$$\chi(\Omega^2_{Gr(2,7)}(n)) = \chi(\wedge^2 T_{Gr(2,7)}(-n) \otimes \mathcal{O}_{Gr(2,7)}(-7)) = \chi(\wedge^2 T_{Gr(2,7)}(-n-7)),$$

then we have

$$\chi(\Omega^2_{Gr(2,7}|x) = \chi(\wedge^2 T_{Gr(2,7)}(-3)) - 4\chi(\wedge^2 T_{Gr(2,7)}(-4)) + 6\chi(\wedge^2 T_{Gr(2,7)}(-5))$$

$$- 4\chi(\wedge^2 T_{Gr(2,7)}(-6)) + \chi(\wedge^2 T_{Gr(2,7)}(-7)).$$

Recall the formula in Section 4.1:

$$\chi(\wedge^2 T_{Gr(2,n)}(m)) = \dim K_{(m+3,m+1,1,\cdots,1,0,0)}V + \dim K_{(m+3,m+3,2,\cdots,2,0)}V$$

then

$$\begin{aligned}
\chi(\wedge^2 T_{Gr(2,7)}(m)) &= \dim K_{(m+3,m+1,1,1,1,0,0)}V + \dim K_{(m+3,m+3,2,2,2,2,0)}V \\
&= \frac{3}{1}\cdot\frac{m+4}{2}\cdot\frac{m+5}{3}\cdot\frac{m+6}{4}\cdot\frac{m+8}{5}\cdot\frac{m+9}{6} \\
&\quad \cdot\frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5} \\
&\quad \cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{2}{1}\cdot\frac{3}{2} \\
&+\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5}\cdot\frac{m+9}{6} \\
&\quad \cdot\frac{m+2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+8}{5} \\
&\quad \cdot\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}\cdot\frac{6}{4}.
\end{aligned}$$

Plugging in the corresponding m's, we eventually are left with

$$\begin{aligned}
\chi(\Omega^2_{Gr(2,7)}|_X) &= 0-0+0-0+\chi(\wedge^2 T_{Gr(2,7)}(-7)) \\
&= 1+1 \\
&= 2.
\end{aligned}$$

All above induces that $\chi(\Omega^2_X) = \chi(\Omega^2_{Gr(2,7)}|_X) - 4\chi(\Omega^1_{Gr(2,7)}|_X) + 10\chi(\mathcal{O}_X(-2)) = 2-0+0 = 2$. On the other hand, $\chi(\Omega^2_X) = h^{2,0} - h^{2,1} + h^{2,2} - h^{2,3} + h^{2,4} = 0-0+2-0+h^{2,4} = 2$, that is how we demonstrate that $h^{2,4}$ has to be 0.

To compute $\chi(\Omega^3_X)$

We look at the exact sequence $0 \longrightarrow T_X \longrightarrow T_{Gr(2,7)} \longrightarrow N \longrightarrow 0$; we try to compute all Chern classes of $T_{Gr(2,7)}$ and the normal bundle N of X in $Gr(2,7)$.

Step I: Compute Chern classes of $T_{Gr(2,7)}$

The i-th Chern class is an obstruction to the existence of $(n-i+1)$ everywhere complex linearly independent vector fields on that vector bundle. The i-th Chern class is in the (2i)-th cohomology group of the base space. Let $\mathcal{E}$ be a locally free sheaf. Then the total Chern class of $\mathcal{E}$ is denoted by $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots$, where $c_i(\mathcal{E})$ is the i-th Chern class. The Chern classes of a complex manifold are the Chern classes of its tangent bundle. We abuse the notation below, for example, $c(Gr(2,7))$ indeed stands for $c(T_{Gr(2,7)})$.

[18] gives the formula of the total Chern class of $Gr(2,7)$, which is also not hard to compute by ourselves:

$$
\begin{aligned}
c(Gr(2,7)) = 1 &+ 7\sigma_1 + 25\sigma_{11} + 22\sigma_2 + 98\sigma_{21} + 143\sigma_{22} + 42\sigma_3 + 191\sigma_{31} \\
&+ 336\sigma_{32} + 252\sigma_{33} + 57\sigma_4 + 238\sigma_{41} + 406\sigma_{42} + 350\sigma_{43} \\
&+ 140\sigma_{44} + 63\sigma_5 + 217\sigma_{51} + 315\sigma_{52} + 245\sigma_{53} + 105\sigma_{54} \\
&+ 21\sigma_{55}
\end{aligned}
$$

where all the σ's are Schubert cycles in standard notations.

Arranging the Schubert cycles in a triangle as follows will make the rest calculations much more convenient:

$$
\begin{array}{llllll}
1 & & & & & \\
\sigma_1 & \sigma_{11} & & & & \\
\sigma_2 & \sigma_{21} & \sigma_{22} & & & \\
\sigma_3 & \sigma_{31} & \sigma_{32} & \sigma_{33} & & \\
\sigma_4 & \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \\
\sigma_5 & \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55}
\end{array}
$$

If we only write down the coefficients of the corresponding Schubert cycles, we have

$$c(Gr(2,7)) = \begin{bmatrix} 1 & & & & & \\ 7 & 25 & & & & \\ 22 & 98 & 143 & & & \\ 42 & 191 & 336 & 252 & & \\ 57 & 238 & 406 & 350 & 140 & \\ 63 & 217 & 315 & 245 & 105 & 21 \end{bmatrix}$$

Step II: Compute Chern classes of N

We know that $N = \mathcal{O}(1)^{\oplus 4}$. Observing that a section of $\mathcal{O}(1)$ on $Gr(2,7)$ is the Schubert cycle σ_1, we have

$$c(N) = c(\mathcal{O}(1))^4 = (1+\sigma_1)^4.$$

Therefore, the total Chern class of X can be written as

$$c(X) = \frac{c(Gr(2,7))}{c(N)} = \frac{c(Gr(2,7))}{(1+\sigma_1)^4}.$$

We will divide the 4 copies of $1+\sigma_1$ one by one.

Step III: the division process

We recall Pieri's formula

$$\sigma_{a,b} \cdot \sigma_1 = \sigma_{a+1,b} + \sigma_{a,b+1}.$$

By using the triangle notation as above, we have the following results:

$$\frac{c(Gr(2,7))}{(1+\sigma_1)} = \begin{bmatrix} 1 & & & & & \\ 6 & 19 & & & & \\ 16 & 63 & 80 & & & \\ 26 & 102 & 154 & 98 & & \\ 31 & 105 & 147 & 105 & 35 & \\ 32 & 80 & 88 & 52 & 18 & 3 \end{bmatrix}$$

$$\frac{c(Gr(2,7))}{(1+\sigma_1)^2} = \begin{bmatrix} 1 & & & & & \\ 5 & 14 & & & & \\ 11 & 38 & 42 & & & \\ 15 & 49 & 63 & 35 & & \\ 16 & 40 & 44 & 26 & 9 & \\ 16 & 24 & 20 & 6 & 3 & 0 \end{bmatrix}$$

$$\frac{c(Gr(2,7))}{(1+\sigma_1)^3} = \begin{bmatrix} 1 & & & & & \\ 4 & 10 & & & & \\ 7 & 21 & 21 & & & \\ 8 & 20 & 22 & 13 & & \\ 8 & 12 & 10 & 3 & 6 & \\ 8 & 4 & 6 & -3 & 0 & 0 \end{bmatrix}$$

$$\frac{c(Gr(2,7))}{(1+\sigma_1)^4} = \begin{bmatrix} 1 & & & & & \\ 3 & 7 & & & & \\ 4 & 10 & 11 & & & \\ 4 & 6 & 5 & 8 & & \\ 4 & 2 & 3 & -8 & * & \\ 4 & -2 & * & * & * & * \end{bmatrix}$$

(Note: "*" represents an irrelevant number.)

So we have obtained the total Chern class of X. More precisely, we have:

$$\begin{aligned}
c_0(X) &= 1\\
c_1(X) &= 3\sigma_1\\
c_2(X) &= 4\sigma_2 + 7\sigma_{11}\\
c_3(X) &= 4\sigma_3 + 10\sigma_{21}\\
c_4(X) &= 4\sigma_4 + 6\sigma_{31} + 11\sigma_{22}\\
c_5(X) &= 4\sigma_5 + 2\sigma_{41} + 5\sigma_{32}\\
c_6(X) &= -2\sigma_{51} + 3\sigma_{42} + 8\sigma_{33}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\chi(X) &= c_6(X) \cdot [X]\\
&= (-2\sigma_{51} + 3\sigma_{42} + 8\sigma_{33}) \cdot \sigma_1^4[Gr(2,7)]\\
&= (\sigma_{52} + 11\sigma_{43}) \cdot \sigma_1^3[Gr(2,7)]\\
&= (12\sigma_{53} + 11\sigma_{44}) \cdot \sigma_1^2[Gr(2,7)]\\
&= 23\sigma_{54} \cdot \sigma_1[Gr(2,7)]\\
&= 23\sigma_{55}[Gr(2,7)]\\
&= 23,
\end{aligned}$$

and we conclude that $h^{3,3} = 15$, which completes the Hodge diamond of $X_{4,7}$.

Remark: We use the above Schubert calculus method to compute the Euler number of $X_{6,6}$ and expect the answer to be $\chi(X) = 24$ as shown in Section 4.2.1.

The total Chern class $c(X) = \frac{c(Gr(2,6))}{(1+\sigma_1)^6}$, where

$$c(Gr(2,6)) = \begin{bmatrix} 1 & & & & \\ 6 & 18 & & & \\ 16 & 58 & 67 & & \\ 26 & 91 & 120 & 65 & \\ 31 & 90 & 105 & 60 & 15 \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)} = \begin{bmatrix} 1 & & & & \\ 5 & 13 & & & \\ 11 & 34 & 33 & & \\ 15 & 42 & 45 & 20 & \\ 16 & 32 & 28 & 12 & 3 \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^2} = \begin{bmatrix} 1 & & & & \\ 4 & 9 & & & \\ 7 & 18 & 15 & & \\ 8 & 16 & 14 & 6 & \\ 8 & 8 & 6 & 0 & 3 \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^3} = \begin{bmatrix} 1 & & & & \\ 3 & 6 & & & \\ 4 & 8 & 7 & & \\ 4 & 4 & 3 & 3 & \\ 4 & 0 & 3 & -6 & 9 \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^4} = \begin{bmatrix} 1 & & & & \\ 2 & 4 & & & \\ 2 & 2 & 5 & & \\ 2 & 0 & -2 & 5 & \\ 2 & -2 & 7 & -18 & 27 \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^5} = \begin{bmatrix} 1 & & & & \\ 1 & 3 & & & \\ 1 & -2 & 7 & & \\ 1 & 1 & -10 & 15 & \\ 1 & * & * & * & * \end{bmatrix}$$

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^6} = \begin{bmatrix} 1 & & & & \\ 0 & 3 & & & \\ 1 & -6 & 13 & & \\ * & * & * & * & \\ * & * & * & * & * \end{bmatrix}$$

Thus

$$
\begin{aligned}
c_0(X) &= 1 \\
c_1(X) &= 0 \\
c_2(X) &= \sigma_2 + 3\sigma_{11}.
\end{aligned}
$$

We know $X_{6,6}$ is a $K3$ surface, so $c_1(X) = 0$ is exactly what we want to see.

Then the topological Euler number

$$
\begin{aligned}
\chi(X) &= c_2(X) \cdot [X] \\
&= (\sigma_2 + 3\sigma_{11}) \cdot \sigma_1^6 \cdot [Gr(2,6)] \\
&= (\sigma_3 + 4\sigma_{21}) \cdot \sigma_1^5 \cdot [Gr(2,6)] \\
&= (\sigma_4 + 5\sigma_{31} + 4\sigma_{22}) \cdot \sigma_1^4 \cdot [Gr(2,6)] \\
&= (6\sigma_{41} + 9\sigma_{32}) \cdot \sigma_1^3 \cdot [Gr(2,6)] \\
&= (15\sigma_{42} + 9\sigma_{33}) \cdot \sigma_1^2 \cdot [Gr(2,6)] \\
&= 24\sigma_{43} \cdot \sigma_1 \cdot [Gr(2,6)] \\
&= 24\sigma_{44} \cdot [Gr(2,6)] \\
&= 24
\end{aligned}
$$

is also as desired.

$\mathbf{X_{4,5}}$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 5 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The above Hodge diamond matches with our previous example in Section 2.2, saying $X_{4,5}$ is a Del Pezzo surface of degree 5. It is fairly easy to compute this one, since the Hodge diamond only has one unknown number:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & h^{1,1} & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

To compute $\chi(\Omega_X^1)$, we start with the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,5)}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,5)}(-5) \longrightarrow \mathcal{O}_{Gr(2,5)}(-4)^{\oplus 4} \longrightarrow \mathcal{O}_{Gr(2,5)}(-3)^{\oplus 6} \longrightarrow \mathcal{O}_{Gr(2,5)}(-2)^{\oplus 4}$$
$$\longrightarrow \mathcal{O}_{Gr(2,5)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0.$$

$\chi(\mathcal{O}_{Gr(2,5)}(-5)) = 1, \chi(\mathcal{O}_{Gr(2,5)}(-4)) = \chi(\mathcal{O}_{Gr(2,5)}(-3)) = \chi(\mathcal{O}_{Gr(2,5)}(-2)) = \chi(\mathcal{O}_{Gr(2,5)}(-1)) = 0$ together imply $\chi(\mathcal{O}_X(-1)) = 1$.

Resolve $\Omega^1_{Gr(2,5)}|_X$ to get:

$$0 \longrightarrow \Omega^1_{Gr(2,5)}(-4) \longrightarrow \Omega^1_{Gr(2,5)}(-3)^{\oplus 4} \longrightarrow \Omega^1_{Gr(2,5)}(-2)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,5)}(-1)^{\oplus 4}$$

$$\longrightarrow \Omega^1_{Gr(2,5)} \longrightarrow \Omega^1_{Gr(2,5)}|_X \longrightarrow 0$$

$\chi(\Omega^1_{Gr(2,5)}|_X) = -1$ follows from $\chi(\Omega^1_{Gr(2,5)}(-4)) = \chi(T_{Gr(2,5)}(-1)) = 0$, $\chi(\Omega^1_{Gr(2,5)}(-3)) = \chi(T_{Gr(2,5)}(-2)) = 0$, $\chi(\Omega^1_{Gr(2,5)}(-2)) = \chi(T_{Gr(2,5)}(-3)) = 0$, $\chi(\Omega^1_{Gr(2,5)}(-1)) = \chi(T_{Gr(2,5)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,5)}) = \chi(T_{Gr(2,5)}(-5)) = -1$. Therefore $\chi(\Omega^1_X) = -4 \cdot 1 + (-1) = -5$, and hence $h^{1,1} = 5$.

$\mathbf{X_{4,6}}$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 8 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Since $\dim X = 4$, by the Leftschetz hyperplane theorem, $h^{p,q}(X) = h^{p,q}(Gr(2,6))$ for $p + q \leq 3$. Also note that $K_X = \mathcal{O}(-2)|_X$, i.e., X is Fano, thus the full Hodge diamond looks like:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Now we are left to compute $h^{3,1} = h^{1,3}$ and $h^{2,2}$.

To compute $h^{3,1} = h^{1,3}$, we only need to compute $\chi(\Omega^1_X)$.

We apply our standard machinery

$$0 \to \mathcal{O}_{Gr(2,6)}(-1)^{\oplus 4}|_X \to \Omega^1_{Gr(2,6)}|_X \to \Omega^1_X \to 0$$

In this case, we only need to figure out $\chi(\mathcal{O}_{Gr(2,6)}(-1)|_X)$ and $\chi(\Omega^1_{Gr(2,6)}|_X)$.

Using the Koszul resolution:

$$0 \to \mathcal{O}_{Gr(2,6)}(-5) \to \mathcal{O}_{Gr(2,6)}(-4)^{\oplus 4} \to \mathcal{O}_{Gr(2,6)}(-3)^{\oplus 6} \to \mathcal{O}_{Gr(2,6)}(-2)^{\oplus 4}$$
$$\to \mathcal{O}_{Gr(2,6)}(-1) \to \mathcal{O}_{Gr(2,6)}(-1)|_X \to 0$$

and the following character formula

$$\begin{aligned}
\chi(\mathcal{O}(m)) &= \dim K_{(m,m,0,0,0,0)}V \\
&= \frac{m+2}{2}\frac{m+3}{3}\frac{m+4}{4}\frac{m+5}{5}\frac{m+1}{1}\frac{m+2}{2}\frac{m+3}{3}\frac{m+4}{4},
\end{aligned}$$

we conclude that $\chi(\mathcal{O}_{Gr(2,6)}(-1)|_X) = 0 - 0 + 0 - 0 + 0 = 0$.

On the other hand, the following complex

$$0 \to \Omega^1_{Gr(2,6)}(-4) \to \Omega^1_{Gr(2,6)}(-3)^{\oplus 4} \to \Omega^1_{Gr(2,6)}(-2)^{\oplus 6} \to \Omega^1_{Gr(2,6)}(-1)^{\oplus 4}$$
$$\to \Omega^1_{Gr(2,6)} \to \Omega^1_{Gr(2,6)}|_X \to 0$$

is the same as

$$0 \to T_{Gr(2,6)}(-2) \to T_{Gr(2,6)}(-3)^{\oplus 4} \to T_{Gr(2,6)}(-4)^{\oplus 6} \to T_{Gr(2,6)}(-5)^{\oplus 4}$$
$$\to T_{Gr(2,6)}(-6) \to \Omega^1_{Gr(2,6)}|_X \to 0$$

We recall the following Weyl character formula which we have seen before:

$$\chi(T_{Gr(2,6)}(m)) = \dim K_{(m+2,m+1,1,1,1,0)}V$$
$$= \frac{2}{1} \cdot \frac{m+3}{2} \cdot \frac{m+4}{3} \cdot \frac{m+5}{4} \cdot \frac{m+7}{5} \cdot \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} \cdot \frac{m+5}{4} \cdot \frac{4}{1}$$

Therefore, $\chi(\Omega^1_{Gr(2,6)}|_X) = \chi(T_{Gr(2,6)}(-6)) = -8 \cdot \frac{3\cdot 2\cdot 1\cdot 1}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{5\cdot 4\cdot 3\cdot 1}{1\cdot 2\cdot 3\cdot 4} = -1$.

Combining all the results above, we have $\chi(\Omega^1_X) = -1$, and hence $h^{3,1} = h^{1,3} = 0$.

There are two ways to compute $h^{2,2}$; we will present both of them.

Approach I: To compute $\chi(X)$

From the computation of the previous case, we had

$$\frac{c(Gr(2,6))}{(1+\sigma_1)^4} = \begin{bmatrix} 1 & & & & \\ 2 & 4 & & & \\ 2 & 2 & 5 & & \\ 2 & 0 & -2 & 5 & \\ 2 & -2 & 7 & -18 & 27 \end{bmatrix}$$

We can read: $c(X) = 2\sigma_4 + 5\sigma_{2,2}$, so

$$\begin{aligned}
\chi(X) &= (2\sigma_4 + 5\sigma_{2,2}) \cdot \sigma_1^4[Gr(2,6)] \\
&= (2\sigma_{4,1} + 5\sigma_{3,2}) \cdot \sigma_1^3[Gr(2,6)] \\
&= (7\sigma_{4,2} + 5\sigma_{3,3}) \cdot \sigma_1^2[Gr(2,6)] \\
&= 12\sigma_{4,3} \cdot \sigma_1[Gr(2,6)] \\
&= 12,
\end{aligned}$$

which induces $h^{2,2} = 8$.

Approach II: To compute $\chi(\Omega_X^2)$

Using the following resolution

$$0 \longrightarrow \mathcal{O}_X(-2)^{\oplus 10} \longrightarrow \Omega_{Gr(2,6)}(-1)^{\oplus 4}|_X \longrightarrow \Omega^2_{Gr(2,6)}|_X \longrightarrow \Omega_X^2 \longrightarrow 0,$$

and then by Koszul and above formula of $\chi(\mathcal{O}_{Gr(2,6)}(m))$, we have

$$\begin{aligned}
\chi(\mathcal{O}_X(-2)) &= \chi(\mathcal{O}(-6)) - 4\chi(\mathcal{O}(-5)) + 6\chi(\mathcal{O}(-4)) - 4\chi(\mathcal{O}(-3)) + \chi(\mathcal{O}(-2)) \\
&= 1 - 0 + 0 - 0 + 0 \\
&= 1.
\end{aligned}$$

Similarly, by Koszul and above formula of $\chi(\Omega^1_{Gr(2,6)}(m))$, we have

$$\begin{aligned}\chi(\Omega_{Gr(2,6)}(-1)|_X) &= \chi(\Omega(-5)) - 4\chi(\Omega(-4)) + 6\chi(\Omega(-3)) - 4\chi(\Omega(-2)) + \chi(\Omega(-1)) \\ &= \chi(T(-1)) - 4\chi(T(-2)) + 6\chi(T(-3)) - 4\chi(T(-4)) + \chi(T(-5)) \\ &= 0 - 0 + 0 - 0 + 0 \\ &= 0.\end{aligned}$$

Finally by Koszul we get

$$\begin{aligned}\chi(\Omega^2_{Gr(2,6)}|_X) &= \chi(\Omega^2(-4)) - 4\chi(\Omega^2(-3)) + 6\chi(\Omega^2(-2)) - 4\chi(\Omega^2(-1)) + \chi(\Omega^2) \\ &= \chi(\wedge^2 T(-2)) - 4\chi(\wedge^2 T(-3)) + 6\chi(\wedge^2 T(-4)) - 4\chi(\wedge^2 T(-5)) + \chi(\wedge^2 T(-6)).\end{aligned}$$

The dimension formula has two terms:

$$\begin{aligned}\chi(\wedge^2 T_{Gr(2,6)}(m)) &= \dim K_{(m+3,m+1,1,1,0,0)}V + \dim K_{(m+3,m+3,2,2,2,0)}V \\ &= 18 \cdot \frac{(m+4)(m+5)(m+7)(m+8)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+1)(m+2)(m+4)(m+5)}{1\cdot 2\cdot 3\cdot 4} \\ &+ 10 \cdot \frac{(m+3)(m+4)(m+5)(m+8)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+2)(m+3)(m+4)(m+7)}{1\cdot 2\cdot 3\cdot 4}.\end{aligned}$$

The contribution from the first term is $0-4+0-0+1=-3$; while the contribution from the second term is $0-0+0-0+1=1$; as a result, $\chi(\Omega^2_{Gr(2,6)|X}) = -2$.

Putting every piece together, we eventually end up with $\chi(\Omega^2_X) = 10-0+(-2) = 8$, which also implies $h^{2,2} = 8$.

4.3 Examples in codimension $r = 3$

Next, we will compute the Hodge diamonds of $X_{3,n}$ for $n = 5, 6, 7, 8, 9$ in turn. Before this, we should point out that Theorem 3.1.1 actually covers all these cases in this section. We will still follow our standard machinery to deal with those cases though.

$\mathbf{X_{3,5}}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We first list the Weyl dimension formulas needed in this case:

$$\chi(\mathcal{O}_{Gr(2,5)}(m)) = \frac{(m+2)(m+3)(m+4)}{2 \cdot 3 \cdot 4} \cdot \frac{(m+1)(m+2)(m+3)}{1 \cdot 2 \cdot 3}$$

$$\chi(T_{Gr(2,5)}(m)) = 6 \cdot \frac{(m+3)(m+4)(m+6)}{2 \cdot 3 \cdot 4} \cdot \frac{(m+1)(m+2)(m+4)}{1 \cdot 2 \cdot 3}$$

Example 2.2.2 already showed that $X_{3,5}$ is birational to a quadric in $\mathbb{P}^4$, whose Hodge diamond is a pure type. Alternatively, we can figure the Hodge diamond out by our standard machinery. This is an easy case, because there is only one unknown Hodge number $h^{2,1} = h^{1,2}$ and it suffices to compute $\chi(\Omega^1_X)$. The incomplete Hodge diamond of $X_{3,5}$ is as the following:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & h^{2,1} & & h^{1,2} & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We have the regular short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,5)}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get the Koszul complex:

$$0 \longrightarrow \mathcal{O}_{Gr(2,5)}(-4) \longrightarrow \mathcal{O}_{Gr(2,5)}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,5)}(-2)^{\oplus 3}$$
$$\longrightarrow \mathcal{O}_{Gr(2,5)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0.$$

$\chi(\mathcal{O}_X(-1)) = 0$ comes from $\chi(\mathcal{O}_{Gr(2,5)}(-4)) = \chi(\mathcal{O}_{Gr(2,5)}(-3)) = \chi(\mathcal{O}_{Gr(2,5)}(-2)) = \chi(\mathcal{O}_{Gr(2,5)}(-1)) = 0$, which are directly induced from the characteristic formula we posted at the beginning of this example.

Resolve $\Omega^1_{Gr(2,5)}|_X$ to get another Koszul complex:

$$0 \longrightarrow \Omega^1_{Gr(2,5)}(-3) \longrightarrow \Omega^1_{Gr(2,5)}(-2)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,5)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^1_{Gr(2,5)} \longrightarrow \Omega^1_{Gr(2,5)}|_X \longrightarrow 0$$

$\chi(\Omega^1_{Gr(2,5)}|_X) = -1$ comes from $\chi(\Omega^1_{Gr(2,5)}(-3)) = \chi(T_{Gr(2,5)}(-2)) = 0$, $\chi(\Omega^1_{Gr(2,5)}(-2)) = \chi(T_{Gr(2,5)}(-3)) = 0$, $\chi(\Omega^1_{Gr(2,5)}(-1)) = \chi(T_{Gr(2,5)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,5)}) = \chi(T_{Gr(2,5)}(-5)) = -1$.

In total, $\chi(\Omega^1_X) = (-1) + 3 \cdot 0 = -1$ and hence $h^{1,2} = h^{2,1} = 0$.

$\mathbf{X_{3,6}}$:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 1 & & 0 & & 0 \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

We first summarize formulas we obtained last time:

$$
\begin{aligned}
\chi(\mathcal{O}_{Gr(2,6)}(m)) &= \frac{(m+2)(m+3)(m+4)(m+5)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+1)(m+2)(m+3)(m+4)}{1\cdot 2\cdot 3\cdot 4} \\
\chi(T_{Gr(2,6)}(m)) &= 8 \cdot \frac{(m+3)(m+4)(m+5)(m+7)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+1)(m+2)(m+3)(m+5)}{1\cdot 2\cdot 3\cdot 4} \\
\chi(\wedge^2 T_{Gr(2,6)}(m)) &= 18 \cdot \frac{(m+4)(m+5)(m+7)(m+8)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+1)(m+2)(m+4)(m+5)}{1\cdot 2\cdot 3\cdot 4} \\
&\quad + 10 \cdot \frac{(m+3)(m+4)(m+5)(m+8)}{2\cdot 3\cdot 4\cdot 5} \cdot \frac{(m+2)(m+3)(m+4)(m+7)}{1\cdot 2\cdot 3\cdot 4}
\end{aligned}
$$

Since $\dim Gr(2,6) = 8, \dim X_{3,6} = 5$, then the Hodge diamond looks like:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
0 & & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & & 0 \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

Hence we need to compute $h^{1,4} = h^{4,1}, h^{2,3} = h^{3,2}$.

First we need to compute $\chi(\Omega^1_X)$, and

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,6)} |_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,6)}(-4) \longrightarrow \mathcal{O}_{Gr(2,6)}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,6)}(-2)^{\oplus 3}$$
$$\longrightarrow \mathcal{O}_{Gr(2,6)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0;$$

thus $\chi(\mathcal{O}_X(-1)) = 0$ comes from $\chi(\mathcal{O}_{Gr(2,6)}(-4)) = \chi(\mathcal{O}_{Gr(2,6)}(-3)) = \chi(\mathcal{O}_{Gr(2,6)}(-2)) = \chi(\mathcal{O}_{Gr(2,6)}(-1)) = 0$.

Resolve $\Omega^1_{Gr(2,6)}|_X$ to get:

$$0 \longrightarrow \Omega^1_{Gr(2,6)}(-3) \longrightarrow \Omega_{Gr(2,6)}(-2)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,6)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^1_{Gr(2,6)} \longrightarrow \Omega^1_{Gr(2,6)}|_X \longrightarrow 0,$$

$\chi(\Omega^1_{Gr(2,6)}|_X) = -1$ follows from $\chi(\Omega^1_{Gr(2,6)}(-3)) = \chi(T_{Gr(2,6)}(-3)) = 0$, $\chi(\Omega^1_{Gr(2,6)}(-2)) = \chi(T_{Gr(2,6)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,6)}(-1)) = \chi(T_{Gr(2,6)}(-5)) = 0$, $\chi(\Omega^1_{Gr(2,6)}) = \chi(T_{Gr(2,6)}(-6)) = -1$.

Therefore $\chi(\Omega^1_X) = \chi(\Omega^1_{Gr(2,6)}|_X) - 3 \times \chi(\mathcal{O}_X(-1)) = -1 - 0 = -1$, and hence $h^{1,4} = h^{4,1} = 0$.

Next, we need to compute $\chi(\Omega^2_X)$ and we start with

$$0 \longrightarrow \mathcal{O}_X(-2)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,6)}(-1)^{\oplus 3}|_X \longrightarrow \Omega^2_{Gr(2,6)}|_X \longrightarrow \Omega^2_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-2)$ to get:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{Gr(2,6)}(-5) \longrightarrow \mathcal{O}_{Gr(2,6)}(-4)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,6)}(-3)^{\oplus 3} \\ &\longrightarrow \mathcal{O}_{Gr(2,6)}(-2) \longrightarrow \mathcal{O}_X(-2) \longrightarrow 0. \end{aligned}$$

For the same reason, the Euler characteristics of the first four terms all vanish, which determines

$$\chi(\mathcal{O}_X(-2)) = 0.$$

Similarly, we resolve $\Omega^1_{Gr(2,6)}(-1)|_X$ to get the following Koszul complex:

$$\begin{aligned} 0 &\longrightarrow \Omega^1_{Gr(2,6)}(-4) \longrightarrow \Omega^1_{Gr(2,6)}(-3)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,6)}(-2)^{\oplus 3} \\ &\longrightarrow \Omega^1_{Gr(2,6)}(-1) \longrightarrow \Omega^1_{Gr(2,6)}(-1)|_X \longrightarrow 0. \end{aligned}$$

$\chi(\Omega^1_{Gr(2,6)}(-1)|_X) = 0$ is computed as

$$\chi(\Omega^1_{Gr(2,6)}(-4)) = \chi(T_{Gr(2,6)}(-2)) = 0,$$

$$\chi(\Omega^1_{Gr(2,6)}(-3)) = \chi(T_{Gr(2,6)}(-3)) = 0,$$

$$\chi(\Omega^1_{Gr(2,6)}(-2)) = \chi(T_{Gr(2,6)}(-4)) = 0,$$

$$\chi(\Omega^1_{Gr(2,6)}(-1)) = \chi(T_{Gr(2,6)}(-5)) = 0.$$

Resolve $\Omega^2_{Gr(2,6)}|_X$ to get:

$$0 \longrightarrow \Omega^2_{Gr(2,6)}(-3) \longrightarrow \Omega^2_{Gr(2,6)}(-2)^{\oplus 3} \longrightarrow \Omega^2_{Gr(2,6)}(-1)^{\oplus 3}$$

$$\longrightarrow \Omega^2_{Gr(2,6)} \longrightarrow \Omega^2_{Gr(2,6)}|_X \longrightarrow 0;$$

since

$$\chi(\Omega^2_{Gr(2,6)}(-3)) = \chi(\wedge^2 T_{Gr(2,6)}(-3)) = 1,$$

$$\chi(\Omega^2_{Gr(2,6)}(-2)) = \chi(\wedge^2 T_{Gr(2,6)}(-4)) = 0,$$

$$\chi(\Omega^2_{Gr(2,6)}(-1)) = \chi(\wedge^2 T_{Gr(2,6)}(-5)) = 0,$$

$$\chi(\Omega^2_{Gr(2,6)}) = \chi(\wedge^2 T_{Gr(2,6)}(-6)) = 2,$$

we get $\chi(\Omega^2_{Gr(2,6)}|_X) = 1$.

Therefore $\chi(\Omega^2_X) = \chi(\Omega^2_{Gr(2,6)}|_X) - 3\chi(\Omega^1_{Gr(2,6)}(-1)|_X) + 6\chi(\mathcal{O}_X(-2)) = 1-0+0 = 1$, and hence $h^{3,2} = h^{2,3} = 1$.

$\mathbf{X_{3,7}}$:

$$
\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,. \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}
$$

We first summarize formulas that we need:

$$
\begin{aligned}
\chi(\mathcal{O}_{Gr(2,7)}(m)) &= \frac{(m+2)(m+3)(m+4)(m+5)(m+6)}{2\cdot 3\cdot 4\cdot 5\cdot 6} \\
&\quad \cdot \frac{(m+1)(m+2)(m+3)(m+4)(m+5)}{1\cdot 2\cdot 3\cdot 4\cdot 5} \\
\chi(T_{Gr(2,7)}(m)) &= 10\cdot \frac{(m+3)(m+4)(m+5)(m+6)(m+8)}{2\cdot 3\cdot 4\cdot 5\cdot 6} \\
&\quad \cdot \frac{(m+1)(m+2)(m+3)(m+4)(m+6)}{1\cdot 2\cdot 3\cdot 4\cdot 5} \\
\chi(\wedge^2 T_{Gr(2,7)}(m)) &= 30\cdot \frac{(m+4)(m+5)(m+6)(m+8)(m+9)}{2\cdot 3\cdot 4\cdot 5\cdot 6} \\
&\quad \cdot \frac{(m+1)(m+2)(m+3)(m+5)(m+6)}{1\cdot 2\cdot 3\cdot 4\cdot 5} \\
&\quad + 15\cdot \frac{(m+3)(m+4)(m+5)(m+6)(m+9)}{2\cdot 3\cdot 4\cdot 5\cdot 6} \\
&\quad \cdot \frac{(m+2)(m+3)(m+4)(m+5)(m+8)}{1\cdot 2\cdot 3\cdot 4\cdot 5}
\end{aligned}
$$

Note dim $Gr(2,7) = 10$ and dim $X = 7$ then the Hodge diamond is:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & 0 & & 1 & & 0 & & & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & 2 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 2 & & 0 & & 0 & & 0 & \\
0 & & h^{6,1} & & h^{5,2} & & h^{4,3} & & h^{3,4} & & h^{2,5} & & h^{1,6} & & 0 \\
 & 0 & & 0 & & 0 & & 2 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 2 & & 0 & & 0 & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & & & 0 & & 1 & & 0 & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & & & 1 & & & & & & &
\end{array}
$$

First, we start with the short exact sequence below

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,7)}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,7)}(-4) \longrightarrow \mathcal{O}_{Gr(2,7)}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,7)}(-2)^{\oplus 3}$$

$$\longrightarrow \mathcal{O}_{Gr(2,7)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0,$$

$\chi(\mathcal{O}_X(-1)) = 0$ is derived from

$$\chi(\mathcal{O}(-4)) = \chi(\mathcal{O}_{Gr(2,7)}(-3)) = \chi(\mathcal{O}_{Gr(2,7)}(-2)) = \chi(\mathcal{O}_{Gr(2,7)}(-1)) = 0.$$

Resolve $\Omega^1_{Gr(2,7)}|_X$ to get:

$$0 \longrightarrow \Omega^1_{Gr(2,7)}(-3) \longrightarrow \Omega^1_{Gr(2,7)}(-2)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,7)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^1_{Gr(2,7)} \longrightarrow \Omega^1_{Gr(2,7)}|_X \longrightarrow 0;$$

$\chi(\Omega^1_{Gr(2,7)}|_X) = -1$ follows from $\chi(\Omega^1(-3)) = \chi(T(-4)) = 0$, $\chi(\Omega^1(-2)) = \chi(T(-5)) = 0$, $\chi(\Omega^1(-1)) = \chi(T(-6)) = 0$, $\chi(\Omega^1) = \chi(T(-7)) = -1$.

Therefore $\chi(\Omega^1_X) = -0 + (-1) = -1$, and hence $h^{1,6} = h^{6,1} = 0$.

Then to compute $\chi(\Omega^2_X)$, we look at the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-2)^{\oplus 6} \longrightarrow \Omega^1_{Gr(2,7)}(-1)^{\oplus 3}|_X \longrightarrow \Omega^2_{Gr(2,7)}|_X \longrightarrow \Omega^2_X \longrightarrow 0$$

We first need to resolve $\mathcal{O}_X(-2)$ to yield:

$$0 \longrightarrow \mathcal{O}_{Gr(2,7)}(-5) \longrightarrow \mathcal{O}_{Gr(2,7)}(-4)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,7)}(-3)^{\oplus 3}$$
$$\longrightarrow \mathcal{O}_{Gr(2,7)}(-2) \longrightarrow \mathcal{O}_X(-2) \longrightarrow 0.$$

For the same reason, the Euler characteristics of the first four terms are all 0; we therefore conclude that $\chi(\mathcal{O}_X(-2)) = 0$.

Resolve $\Omega^1_{Gr(2,7)}(-1)|_X$ to get:

$$0 \longrightarrow \Omega^1_{Gr(2,7)}(-4) \longrightarrow \Omega^1_{Gr(2,7)}(-3)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,7)}(-2)^{\oplus 3}$$
$$\longrightarrow \Omega^1_{Gr(2,7)}(-1) \longrightarrow \Omega^1_{Gr(2,7)}(-1)|_X \longrightarrow 0,$$

$\chi(\Omega^1_{Gr(2,7)}(-4)) = \chi(T_{Gr(2,7)}(-3)) = 0$, $\chi(\Omega^1_{Gr(2,7)}(-3)) = \chi(T_{Gr(2,7)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,7)}(-2)) = \chi(T_{Gr(2,7)}(-5)) = 0$ and $\chi(\Omega^1_{Gr(2,7)}(-1)) = \chi(T_{Gr(2,7)}(-6)) = 0$ guarantee that $\chi(\Omega^1_{Gr(2,7)}(-1)|_X) = 0$.

Next, we resolve $\Omega^2_{Gr(2,7)}|_X$ to get:

$$0 \longrightarrow \Omega^2_{Gr(2,7)}(-3) \longrightarrow \Omega^2_{Gr(2,7)}(-2)^{\oplus 3} \longrightarrow \Omega^2_{Gr(2,7)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^2_{Gr(2,7)} \longrightarrow \Omega^2_{Gr(2,7)}|_X \longrightarrow 0;$$

since $\chi(\Omega^2_{Gr(2,7)}(-3)) = \chi(\wedge^2 T_{Gr(2,7)}(-4)) = 0$, $\chi(\Omega^2_{Gr(2,7)}(-2)) = \chi(\wedge^2 T_{Gr(2,7)}(-5)) = 0$, $\chi(\Omega^2_{Gr(2,7)}(-1)) = \chi(\wedge^2 T_{Gr(2,7)}(-6)) = 0$ and $\chi(\Omega^2_{Gr(2,7)}) = \chi(\wedge^2 T_{Gr(2,7)}(-7)) = 2$, we see that $\chi(\Omega^2_{Gr(2,7)}|_X) = 2$.

Summing these three parts together gives $\chi(\Omega^2_X) = 0 - 0 + 2 = 2$, and that leads to $h^{5,2} = h^{2,5} = 0$.

The last thing remaining for this Hodge diamond is $h^{4,3} = h^{3,4}$. We can work this out if we can compute the Euler number of $X_{3,7}$. Recall that we already have all the Chern classes of $X_{3,7}$ in terms of Schubert cycles, this gives us a hint:

$$c(X) = c(T_X) = \frac{c(Gr(2,7))}{c(N)^3} = \frac{c(Gr(2,7))}{(1+\sigma_1)^3}$$

and

$$\frac{c(Gr(2,7))}{(1+\sigma_1)^3} = \begin{bmatrix} 1 & & & & & \\ 4 & 10 & & & & \\ 7 & 21 & 21 & & & \\ 8 & 20 & 22 & 13 & & \\ 8 & 12 & 10 & 3 & 6 & \\ 8 & 4 & 6 & -3 & 0 & 0 \end{bmatrix}$$

indicate

$$c_0(X) = 1$$

$$c_1(X) = 4\sigma_1$$

$$c_2(X) = 7\sigma_2 + 10\sigma_{11}$$

$$c_3(X) = 8\sigma_3 + 21\sigma_{21}$$

$$c_4(X) = 8\sigma_4 + 20\sigma_{31} + 21\sigma_{22}$$

$$c_5(X) = 8\sigma_5 + 12\sigma_{41} + 22\sigma_{32}$$

$$c_6(X) = 4\sigma_{51} + 10\sigma_{42} + 13\sigma_{33}$$

$$c_7(X) = 6\sigma_{52} + 3\sigma_{43}.$$

Therefore, we find

$$\begin{aligned}
\chi(X) &= c_7(X) \cdot [X] \\
&= (6\sigma_{52} + 3\sigma_{43}) \cdot \sigma_1^3[Gr(2,7)] \\
&= (9\sigma_{53} + 3\sigma_{44}) \cdot \sigma_1^2[Gr(2,7)] \\
&= 12\sigma_{54} \cdot \sigma_1[Gr(2,7)] \\
&= 12\sigma_{55}[Gr(2,7)] \\
&= 12.
\end{aligned}$$

Since we already have the following incomplete Hodge diamond

$$\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad h^{4,3} \quad h^{3,4} \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}$$

we find out those Betti numbers:

$$b_0 = b_2 = b_{12} = b_{14} = 1$$

$$b_4 = b_6 = b_8 = b_{10} = 2$$

$$b_1 = b_3 = b_5 = b_9 = b_{11} = b_{13} = 0.$$

On the other hand,

$$\chi(X) = \sum_{i=0}^{14} (-1)^i b_i$$

leaves $b_7 = 0$, which indicates that $h^{3,4} = h^{43} = 0$.

$\mathbf{X_{3,8}}$:

$$\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}$$

In this case, $\dim X = 9$

$$c(Gr(2,8)) = \begin{bmatrix}
1 & & & & & & \\
8 & 33 & & & & & \\
29 & 152 & 266 & & & & \\
64 & 353 & 768 & 731 & & & \\
99 & 528 & 1171 & 1344 & 742 & & \\
120 & 575 & 1176 & 1316 & 840 & 266 & \\
127 & 504 & 868 & 840 & 490 & 168 & 28
\end{bmatrix}$$

$$\frac{c(Gr(2,8))}{(1+\sigma_1)} = \begin{bmatrix} 1 & & & & & & \\ 7 & 26 & & & & & \\ 22 & 104 & 162 & & & & \\ 42 & 207 & 399 & 332 & & & \\ 57 & 264 & 508 & 504 & 238 & & \\ 63 & 248 & 420 & 392 & 210 & 56 & \\ 64 & 192 & 256 & 192 & 88 & 24 & 4 \end{bmatrix}$$

$$\frac{c(Gr(2,8))}{(1+\sigma_1)^2} = \begin{bmatrix} 1 & & & & & & \\ 6 & 20 & & & & & \\ 16 & 68 & 94 & & & & \\ 26 & 113 & 192 & 140 & & & \\ 31 & 120 & 196 & 168 & 70 & & \\ 32 & 96 & 128 & 96 & 44 & 12 & \\ 32 & 64 & 64 & 32 & 12 & 0 & 4 \end{bmatrix}$$

$$\frac{c(Gr(2,8))}{(1+\sigma_1)^3} = \begin{bmatrix} 1 & & & & & & \\ 5 & 15 & & & & & \\ 11 & 42 & 52 & & & & \\ 15 & 56 & 84 & 56 & & & \\ 16 & 48 & 64 & 48 & 22 & & \\ 16 & 32 & 32 & 16 & 6 & 6 & \\ 16 & 16 & 16 & 0 & 6 & -12 & 16 \end{bmatrix}$$

Therefore, $c_9(X) = 6\sigma_{5,4}$.

$$\begin{aligned} \chi(X) &= 6\sigma_{5,4} \cdot \sigma_1^3[Gr(2,8)] \\ &= (6\sigma_{6,4} + 6\sigma_{5,5}) \cdot \sigma_1^2[Gr(2,8)] \\ &= 12\sigma_{6,5} \cdot \sigma_1[Gr(2,8)] \\ &= 12\sigma_{6,6}[Gr(2,8)] \\ &= 12. \end{aligned}$$

Note that by the Lefschetz Hyperplane Theorem,

$$h^0(X) = h^2(X) = h^{16}(X) = h^{18}(X) = 1$$

$$h^4(X) = h^6(X) = h^{12}(X) = h^{14}(X) = 2$$

$$h^8(X) = h^{10}(X) = 3$$

$$h^1(X) = h^3(X) = h^5(X) = h^7(X) = h^{11}(X) = h^{13}(X) = h^{15}(X) = h^{17}(X) = 0.$$

Therefore, $\chi(X) = 12$ gives $h^9(X) = 6$, i.e., there are nontrivial Hodge numbers in this case.

In order to complete the Hodge diamond, we only need to compute the middle row, i.e. $h^{p,q}$ for $p + q = 9$.

First, because X is Fano, $h^{9,0} = h^{0,9} = 0$. Now we summarize all necessary formulas on $Gr(2,8)$,

$$\begin{aligned}\chi(\mathcal{O}_{Gr(2,8)}(m)) = &\frac{m+2}{2} \cdot \frac{m+3}{3} \cdot \frac{m+4}{4} \cdot \frac{m+5}{5} \cdot \frac{m+6}{6} \cdot \frac{m+7}{7} \\ &\cdot \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} \cdot \frac{m+4}{4} \cdot \frac{m+5}{5} \cdot \frac{m+6}{6}\end{aligned}$$

$$\begin{aligned}\chi(T_{Gr(2,8)}(m)) = &\frac{2}{1} \cdot \frac{m+3}{2} \cdot \frac{m+4}{3} \cdot \frac{m+5}{4} \cdot \frac{m+6}{5} \cdot \frac{m+7}{6} \cdot \frac{m+9}{7} \\ &\cdot \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} \cdot \frac{m+4}{4} \cdot \frac{m+5}{5} \cdot \frac{m+7}{6} \cdot \frac{6}{1}\end{aligned}$$

$$\chi(\wedge^2 T_{Gr(2,8)}(m)) = \frac{3}{1}\cdot\frac{m+4}{2}\cdot\frac{m+5}{3}\cdot\frac{m+6}{4}\cdot\frac{m+7}{5}\cdot\frac{m+9}{6}\cdot\frac{m+10}{7}$$
$$\cdot\frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot\frac{m+4}{4}\cdot\frac{m+6}{5}\cdot\frac{m+7}{6}\cdot\frac{5}{1}\cdot\frac{6}{2}$$
$$+\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5}\cdot\frac{m+7}{6}\cdot\frac{m+10}{7}\cdot\frac{m+2}{1}$$
$$\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5}\cdot\frac{m+9}{6}\cdot\frac{6}{1}\cdot\frac{7}{2}$$

$$\chi(\wedge^3 T_{Gr(2,8)}(m)) = \frac{4}{1}\cdot\frac{m+5}{2}\cdot\frac{m+6}{3}\cdot\frac{m+7}{4}\cdot\frac{m+9}{5}\cdot\frac{m+10}{6}\cdot\frac{m+11}{7}$$
$$\cdot\frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot\frac{m+5}{4}\cdot\frac{m+6}{5}\cdot\frac{m+7}{6}\cdot\frac{4}{1}\cdot\frac{5}{2}\cdot\frac{6}{3}$$
$$+\frac{2}{1}\cdot\frac{m+4}{2}\cdot\frac{m+5}{3}\cdot\frac{m+6}{4}\cdot\frac{m+7}{5}\cdot\frac{m+9}{6}\cdot\frac{m+11}{7}$$
$$\cdot\frac{m+2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdot\frac{m+5}{4}\cdot\frac{m+7}{5}\cdot\frac{m+9}{6}$$
$$\cdot\frac{5}{1}\cdot\frac{6}{2}\cdot\frac{7}{3}\cdot\frac{2}{1}$$

To compute $\chi(\Omega^1_X)$

We look at the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus 3} \longrightarrow \Omega^1_{Gr(2,8)}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,8)}(-4) \longrightarrow \mathcal{O}_{Gr(2,8)}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,8)}(-2)^{\oplus 3}$$
$$\longrightarrow \mathcal{O}_{Gr(2,8)}(-1) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0$$

$\chi(\mathcal{O}_X(-1)) = 0$ comes from $\chi(\mathcal{O}_{Gr(2,8)}(-4)) = \chi(\mathcal{O}_{Gr(2,8)}(-3)) = \chi(\mathcal{O}_{Gr(2,8)}(-2)) =$

$\chi(\mathcal{O}_{Gr(2,8)}(-1)) = 0.$

Resolve $\Omega^1_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega_{Gr(2,8)}(-3) \longrightarrow \Omega_{Gr(2,8)}(-2)^{\oplus 3} \longrightarrow \Omega_{Gr(2,8)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega_{Gr(2,8)} \longrightarrow \Omega_{Gr(2,8)}|_X \longrightarrow 0$$

$\chi(\Omega^1_{Gr(2,8)}|_X) = -1$ follows from $\chi(\Omega^1_{Gr(2,8)}(-3)) = \chi(T_{Gr(2,8)}(-5)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-2)) = \chi(T_{Gr(2,8)}(-6)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-1)) = \chi(T_{Gr(2,8)}(-7)) = 0$ and $\chi(\Omega^1_{Gr(2,8)}) = \chi(T_{Gr(2,8)}(-8)) = -1$.

So $\chi(\Omega_X) = \chi(\Omega_{Gr(2,8)}|_X) - 3\chi(\mathcal{O}_X(-1)) = -1$, which induces $h^{8,1} = h^{1,8} = 0$.

To compute $\chi(\Omega^2_X)$

We take the wedge of the short exact sequence in Step I to get:

$$0 \longrightarrow \mathcal{O}_X(-2)^{\oplus 6} \longrightarrow \Omega_{Gr(2,8)}(-1)^{\oplus 3}|_X \longrightarrow \Omega^2_{Gr(2,8)}|_X \longrightarrow \Omega^2_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,8)}(-5) \longrightarrow \mathcal{O}_{Gr(2,8)}(-4)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,8)}(-3)^{\oplus 3}$$
$$\longrightarrow \mathcal{O}_{Gr(2,8)}(-2) \longrightarrow \mathcal{O}_X(-2) \longrightarrow 0$$

$\chi(\mathcal{O}_X(-2)) = 0$ comes from $\chi(\mathcal{O}_{Gr(2,8)}(-5)) = \chi(\mathcal{O}_{Gr(2,8)}(-4)) = \chi(\mathcal{O}_{Gr(2,8)}(-3)) = \chi(\mathcal{O}_{Gr(2,8)}(-2)) = 0.$

Resolve $\Omega^1_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega_{Gr(2,8)}(-4) \longrightarrow \Omega_{Gr(2,8)}(-3)^{\oplus 3} \longrightarrow \Omega_{Gr(2,8)}(-2)^{\oplus 3}$$
$$\longrightarrow \Omega_{Gr(2,8)}(-1) \longrightarrow \Omega_{Gr(2,8)}(-1)|_X \longrightarrow 0$$

$\chi(\Omega^1_{Gr(2,8)}(-1)|_X) = 0$ follows from $\chi(\Omega^1_{Gr(2,8)}(-4)) = \chi(T_{Gr(2,8)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-3)) = \chi(T_{Gr(2,8)}(-5)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-2)) = \chi(T_{Gr(2,8)}(-6)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-1)) = \chi(T_{Gr(2,8)}(-8)) = 0$.

Resolve $\Omega^2_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega^2_{Gr(2,8)}(-3) \longrightarrow \Omega^2_{Gr(2,8)}(-2)^{\oplus 3} \longrightarrow \Omega^2_{Gr(2,8)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^2_{Gr(2,8)} \longrightarrow \Omega^2_{Gr(2,8)}|_X \longrightarrow 0$$

$\chi(\Omega^2_{Gr(2,8)}|_X) = 0$ comes from $\chi(\Omega^2_{Gr(2,8)}(-3)) = \chi(\wedge^2 T_{Gr(2,8)}(-5)) = 0$, $\chi(\Omega^2_{Gr(2,8)}(-2)) = \chi(\wedge^2 T_{Gr(2,8)}(-6)) = 0$, $\chi(\Omega^2_{Gr(2,8)}(-1)) = \chi(\wedge^2 T_{Gr(2,8)}(-7)) = 0$, $\chi(\Omega^2_{Gr(2,8)}) = \chi(\wedge^2 T_{Gr(2,8)}(-8)) = 2$.

Therefore, $\chi(\Omega^2_X) = \chi(\wedge^2 \Omega_{Gr(2,8)}|_X) - 3\chi(\Omega_{Gr(2,8)}(-1)|_X) + 6\chi(\mathcal{O}_X(-2)) = 2 - 0 + 0 = 2$ tells us that $h^{7,2} = h^{2,7} = 0$.

To compute $\chi(\Omega^3_X)$

The exact sequence below is also derived from the first short exact sequence in the first step by taking wedges:

$$0 \longrightarrow \mathcal{O}_X(-3)^{\oplus 10} \longrightarrow \Omega_{Gr(2,8)}(-2)^{\oplus 6}|_X \longrightarrow \Omega^2_{Gr(2,8)}(-1)^{\oplus 3}|_X$$

$$\longrightarrow \Omega^3_{Gr(2,8)}|X \longrightarrow \Omega^3_X \longrightarrow 0.$$

Resolve $\mathcal{O}_X(-1)$ to get:

$$0 \longrightarrow \mathcal{O}_{Gr(2,8)}(-6) \longrightarrow \mathcal{O}_{Gr(2,8)}(-5)^{\oplus 3} \longrightarrow \mathcal{O}_{Gr(2,8)}(-4)^{\oplus 3}$$

$$\longrightarrow \mathcal{O}_{Gr(2,8)}(-3) \longrightarrow \mathcal{O}_X(-3) \longrightarrow 0$$

$\chi(\mathcal{O}_X(-3)) = 0$ comes from $\chi(\mathcal{O}_{Gr(2,8)}(-6)) = \chi(\mathcal{O}_{Gr(2,8)}(-5)) = \chi(\mathcal{O}_{Gr(2,8)}(-4)) = \chi(\mathcal{O}_{Gr(2,8)}(-3)) = 0$.

Resolve $\Omega^1_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega_{Gr(2,8)}(-5) \longrightarrow \Omega_{Gr(2,8)}(-4)^{\oplus 3} \longrightarrow \Omega_{Gr(2,8)}(-3)^{\oplus 3}$$

$$\longrightarrow \Omega_{Gr(2,8)}(-2) \longrightarrow \Omega_{Gr(2,8)}(-2)|_X \longrightarrow 0$$

$\chi(\Omega^1_{Gr(2,8)}(-2)|_X) = 0$ follows from $\chi(\Omega^1_{Gr(2,8)}(-5)) = \chi(\mathcal{T}_{Gr(2,8)}(-3)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-4)) = \chi(\mathcal{T}_{Gr(2,8)}(-4)) = 0$, $\chi(\Omega^1_{Gr(2,8)}(-3)) = \chi(\mathcal{T}_{Gr(2,8)}(-5)) = 0$ and $\chi(\Omega^1_{Gr(2,8)}(-2)) = \chi(\mathcal{T}_{Gr(2,8)}(-6)) = 0$.

Resolve $\Omega^2_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega^2_{Gr(2,8)}(-4) \longrightarrow \Omega^2_{Gr(2,8)}(-3)^{\oplus 3} \longrightarrow \Omega^2_{Gr(2,8)}(-2)^{\oplus 3}$$
$$\longrightarrow \Omega^2_{Gr(2,8)}(-1) \longrightarrow \Omega^2_{Gr(2,8)}(-1)|_X \longrightarrow 0$$

$\chi(\Omega^2_{Gr(2,8)}(-1)|_X) = 0$ follows from $\chi(\Omega^2_{Gr(2,8)}(-4)) = \chi(\wedge^2 T_{Gr(2,8)}(-4)) = 0$, $\chi(\Omega^2_{Gr(2,8)}(-3)) = \chi(\wedge^2 T_{Gr(2,8)}(-5)) = 0$, $\chi(\Omega^2_{Gr(2,8)}(-2)) = \chi(\wedge^2 T_{Gr(2,8)}(-6)) = 0$ and $\chi(\Omega^2_{Gr(2,8)}(-1)) = \chi(\wedge^2 T_{Gr(2,8)}(-7)) = 0$.

Resolve $\Omega^3_{Gr(2,8)}|_X$ to get:

$$0 \longrightarrow \Omega^3_{Gr(2,8)}(-3) \longrightarrow \Omega^3_{Gr(2,8)}(-2)^{\oplus 3} \longrightarrow \Omega^3_{Gr(2,8)}(-1)^{\oplus 3}$$
$$\longrightarrow \Omega^3_{Gr(2,8)} \longrightarrow \Omega^3_{Gr(2,8)}|_X \longrightarrow 0$$

$\chi(\Omega^3_{Gr(2,8)}|_X) = -2$ follows from $\chi(\Omega^3_{Gr(2,8)}(-3)) = \chi(\wedge^3 T_{Gr(2,8)}(-5)) = 0$, $\chi(\Omega^3_{Gr(2,8)}(-2)) = \chi(\wedge^3 T_{Gr(2,8)}(-6)) = 0$, $\chi(\Omega^3_{Gr(2,8)}(-1)) = \chi(\wedge^3 T_{Gr(2,8)}(-7)) = 0$ and $\chi(\Omega^3_{Gr(2,8)}) = \chi(\wedge^3 T_{Gr(2,8)}(-8)) = -2$.

Therefore, $\chi(\Omega^3_X) = \chi(\Omega^3_{Gr(2,8)}|_X) - 3\chi(\Omega^2_{Gr(2,8)}(-1)|_X) + 6\chi(\Omega_{Gr(2,8)}(-2)|_X) - 10\chi(\mathcal{O}_X(-3)) = -2 - 0 + 0 - 0 = -2$ tells us that $h^{6,3} = h^{3,6} = 0$.

With all these results combining with the fact $b^2(X) = 6$, it is suffice to see $h^{4,5} = h^{5,4} = 3$.

$\mathbf{X_{3,9}}$:

$$
\begin{array}{c}
1 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 2 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 0 \\
1
\end{array}
$$

In this case, $\dim X = 11$

$$
c(Gr(2,9)) = \begin{bmatrix}
1 & & & & & & & \\
9 & 42 & & & & & & \\
37 & 222 & 451 & & & & & \\
93 & 598 & 1539 & 1765 & & & & \\
163 & 1044 & 2820 & 4014 & 2817 & & & \\
219 & 1322 & 3450 & 5007 & 4242 & 1848 & & \\
247 & 1326 & 3123 & 4200 & 3486 & 1764 & 462 & \\
255 & 1143 & 2268 & 2604 & 1890 & 882 & 252 & 36
\end{bmatrix}
$$

$$
\frac{c(Gr(2,9))}{(1+\sigma_1)} = \begin{bmatrix}
1 & & & & & & & \\
8 & 34 & & & & & & \\
29 & 159 & 292 & & & & & \\
64 & 375 & 872 & 893 & & & & \\
99 & 570 & 1378 & 1743 & 1074 & & & \\
120 & 632 & 1440 & 1824 & 1344 & 504 & & \\
127 & 567 & 1116 & 1260 & 882 & 378 & 84 & \\
128 & 448 & 704 & 640 & 368 & 136 & 32 & 4
\end{bmatrix}
$$

$$\frac{c(Gr(2,9))}{(1+\sigma_1)^2} = \begin{bmatrix} 1 &&&&&&& \\ 7 & 27 &&&&&& \\ 22 & 110 & 182 &&&&& \\ 42 & 223 & 467 & 426 &&&& \\ 57 & 290 & 621 & 696 & 378 &&& \\ 63 & 279 & 540 & 588 & 378 & 126 && \\ 64 & 224 & 352 & 320 & 184 & 68 & 16 & \\ 64 & 160 & 192 & 128 & 56 & 12 & 4 & 0 \end{bmatrix}$$

$$\frac{c(Gr(2,9))}{(1+\sigma_1)^3} = \begin{bmatrix} 1 &&&&&&& \\ 6 & 21 &&&&&& \\ 16 & 73 & 109 &&&&& \\ 26 & 124 & 234 & 192 &&&& \\ 31 & 135 & 252 & 252 & 126 &&& \\ 32 & 112 & 176 & 160 & 92 & 34 && \\ 32 & 80 & 96 & 64 & 28 & 6 & * & \\ 32 & 48 & 48 & 16 & 12 & -6 & * & * \end{bmatrix}$$

Therefore, $c_{11}(X) = 12\sigma_{7,4} + 6\sigma_{6,5}$

$$\begin{aligned} \chi(X) &= (12\sigma_{7,4} + 6\sigma_{6,5}) \cdot \sigma_1^3[Gr(2,9)] \\ &= (18\sigma_{7,5} + 6\sigma_{6,6}) \cdot \sigma_1^2[Gr(2,9)] \\ &= 24\sigma_{7,6} \cdot \sigma_1[Gr(2,9)] \\ &= 24\sigma_{7,7}[Gr(2,8)] \\ &= 24. \end{aligned}$$

Then again, by the Lefschetz Hyperplane Theorem,

$$h^0(X) = h^2(X) = h^{20}(X) = h^{22}(X) = 1$$

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